Numerical Analysis

A new finite volume scheme for anisotropic diffusion problems on general grids: convergence analysis

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Abstract

We introduce here a new finite volume scheme which was developed for the discretization of anisotropic diffusion problems; the originality of this scheme lies in the fact that we are able to prove its convergence under very weak assumptions on the discretization mesh.

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1. Introduction

The scope of this Note is the discretization by a finite volume method of anisotropic diffusion problems on general meshes. Let $\Omega$ be a polygonal (or polyhedral) open subset of $\mathbb{R}^d$ ($d = 2$ or $3$); let $\mathcal{M}_d(\mathbb{R})$ be the set of $d \times d$ symmetric matrices. We consider the following elliptic conservation equation:

\[-\text{div}(\Lambda \nabla u) = f \quad \text{in} \quad \Omega,\]

with boundary condition

\[u = 0 \quad \text{on} \quad \partial \Omega\]

with the following hypotheses on the data:

$\Lambda$ is a measurable function from $\Omega$ to $\mathcal{M}_d(\mathbb{R})$, and there exist $\underline{\lambda}$ and $\overline{\lambda}$ such that $0 < \underline{\lambda} \leq \lambda$ and $\text{Sp}(\Lambda(x)) \subset [\underline{\lambda}, \overline{\lambda}]$ for a.e. $x \in \Omega$. The function $f$ is such that $f \in L^2(\Omega)$.

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In (3), $\text{Sp}(B)$ denotes for all $B \in \mathcal{M}_d(\mathbb{R})$ the set of the eigenvalues of $B$. We consider the following weak formulation of problem (1):

$$\begin{cases}
  u \in H^1_0(\Omega), \\
  \int_{\Omega} \Lambda(x) \nabla u(x) \cdot \nabla v(x) \, dx = \int_{\Omega} f(x) v(x) \, dx, \quad \forall v \in H^1_0(\Omega).
\end{cases}$$

(4)

2. Discrete functional tools

A finite volume discretization of $\Omega$ is a triplet $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P})$, where:

- $\mathcal{M}$ is a finite family of non-empty convex open disjoint subsets of $\Omega$ (the “control volumes”) such that $\overline{\Omega} = \bigcup_{K \in \mathcal{M}} \overline{K}$. For any $K \in \mathcal{M}$, let $\partial K = \overline{K} \setminus K$ be the boundary of $K$ and $m_K > 0$ denote the measure of $K$.
- $\mathcal{E}$ is a finite family of disjoint subsets of $\overline{\Omega}$ (the “edges” of the mesh), such that, for all $\sigma \in \mathcal{E}$, $\sigma$ is a non-empty closed subset of a hyperplane of $\mathbb{R}^d$, which has a measure $m_\sigma > 0$ for the $(d - 1)$-dimensional measure of $\sigma$.
- We assume that, for all $K \in \mathcal{M}$, there exists a subset $\bar{\mathcal{E}}_K$ of $\mathcal{E}$ such that $\partial K = \bigcup_{\sigma \in \mathcal{E}_K} \partial \sigma$. We then denote by $\mathcal{M}_\sigma = \{K \in \mathcal{M}, \sigma \in \mathcal{E}_K\}$. We then assume that, for all $\sigma \in \mathcal{E}$, either $\mathcal{M}_\sigma$ has exactly one element and then $\sigma \subset \partial \Omega$ (boundary edge) or $\mathcal{M}_\sigma$ has exactly two elements (interior edge). For all $\sigma \in \mathcal{E}$, we denote by $x_\sigma$ the barycenter of $\sigma$.
- $\mathcal{P}$ is a family of points of $\Omega$ indexed by $\mathcal{M}$, denoted by $\mathcal{P} = (x_K)_{K \in \mathcal{M}}$, such that $x_K \in K$ and $K$ is star-shaped with respect to $x_K$.

The following notations are used. The size of the discretization is defined by: $h_\mathcal{D} = \sup\{\text{diam}(K), K \in \mathcal{M}\}$. For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}_K$, we denote for a.e. $x \in \sigma$ by $n_{K,\sigma}$ the unit normal vector to $\sigma$ outward to $K$. We denote by $d_{K,\sigma}$ the Euclidean distance between $x_K$ and $\sigma$. The set of interior (resp. boundary) edges is denoted by $\mathcal{E}_\text{int}$ (resp. $\mathcal{E}_\text{ext}$), that is $\mathcal{E}_\text{int} = \{\sigma \in \mathcal{E}; \sigma \not\subset \partial \Omega\}$ (resp. $\mathcal{E}_\text{ext} = \{\sigma \in \mathcal{E}; \sigma \subset \partial \Omega\}$). The regularity of the mesh is measured through the parameter

$$\theta_\mathcal{D} = \min \left\{ \frac{\min(d_{K,\sigma}, d_{L,\sigma})}{\max(d_{K,\sigma}, d_{L,\sigma})}, \sigma \in \mathcal{E}_\text{int}, \mathcal{M}_\sigma = \{K, L\} \right\}.$$

A family $\mathcal{F}$ of discretizations is regular if there exists $\theta > 0$ such that for any $\mathcal{D} \in \mathcal{F}$, $\theta_\mathcal{D} \geq \theta$.

Let $X_\mathcal{D} = \mathbb{R}^{|\mathcal{M}|} \times \mathbb{R}^{|\mathcal{E}|}$ be the set of all $u := ((u_K)_{K \in \mathcal{M}}, (u_\sigma)_{\sigma \in \mathcal{E}})$, and let $X_{\mathcal{D},0} \subset X_\mathcal{D}$ be defined as the set of all $u \in X_\mathcal{D}$ such that $u_\sigma = 0$ for all $\sigma \in \mathcal{E}_\text{ext}$. The space $X_{\mathcal{D},0}$ is equipped with a Euclidean structure, defined by the following inner product:

$$\forall (v, w) \in (X_{\mathcal{D},0})^2, \quad [v, w]_\mathcal{D} = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} \frac{m_\sigma}{d_{K,\sigma}} (v_\sigma - v_K)(w_\sigma - w_K)$$

(5)

and the associated norm: $\|u\|_{1,\mathcal{D}} = ([u, u]_\mathcal{D})^{1/2}$. Let $H_M(\Omega) \subset L^2(\Omega)$ be the set of piecewise constant functions on the control volumes on the mesh $\mathcal{M}$ which is equipped with the following inner norm: $\|u\|_{1,\mathcal{M}} = \inf\{\|v\|_{1,\mathcal{D}}; v \in X_{\mathcal{D},0}, P_{\mathcal{M}}v = u\}$, where for all $u \in X_{\mathcal{D}}$, we define by $P_{\mathcal{M}}u \in H_M(\Omega)$ the element defined by the values $(u_K)_{K \in \mathcal{M}}$ (we then easily see that this definition of $\|\cdot\|_{1,\mathcal{M}}$ coincides with that given in [1] in the case where we set $d_{K,L} = d_{K,\sigma} + d_{L,\sigma}$ for all $\sigma \in \mathcal{E}_\text{int}$ with $\mathcal{M}_\sigma = \{K, L\}$). For all $\varphi \in C(\Omega, \mathbb{R})$, we denote by $P_{\mathcal{D}}(\varphi)$ the element of $X_{\mathcal{D}}$ defined by $((\varphi(x_K))_{K \in \mathcal{M}}, (\varphi(x_\sigma))_{\sigma \in \mathcal{E}})$.

3. The finite volume scheme and its convergence analysis

The finite volume method is based on the discretization of the balance equation associated to Eq. (1) on cell $K$. It requires the definition of consistent numerical fluxes $(F_{K,\sigma}^\mathcal{D})_{K \in \mathcal{M}, \sigma \in \mathcal{E}_K}$ on the edges of the cells, meant to approximate the diffusion fluxes $-\Lambda \nabla u \cdot n_K$, where $n_K$ is the unit outward normal to $\partial K$. 

Let $\mathcal{F}$ be a family of finite volume discretizations; for $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P}) \in \mathcal{F}$, $K \in \mathcal{M}$ and $\sigma \in \mathcal{E}$, we denote by $F_{K,\sigma}^D$ a linear mapping from $X_D$ to $\mathbb{R}^\mathcal{E}$. The family $((F_{K,\sigma}^D)_{K \in \mathcal{M}})_{\sigma \in \mathcal{E}}$ is said to be a consistent family of fluxes if for any function $\varphi \in C^2(\mathbb{R}^d, \mathbb{R})$, 

$$\lim_{h_D \to 0} \max_{K \in \mathcal{M}} \frac{1}{m_K} \left| F_{K,\sigma}^D(\varphi) \right| + \int_{\sigma} A_K \nabla \varphi \cdot n_{K,\sigma} \, dy = 0, \quad \text{(6)}$$

where $A_K = \frac{1}{m_K} \int_K \Lambda \, dx$. In order to get some estimates on the approximate solutions, we need a coercivity property: the family of numerical fluxes $((F_{K,\sigma}^D)_{K \in \mathcal{M}})_{\sigma \in \mathcal{E}}$ is said to be coercive if there exists $\alpha > 0$ such that, for any $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P}) \in \mathcal{F}$ and for any $u \in X_{D,0}$,

$$\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}_K} (u_K - u_\sigma) F_{K,\sigma}^D(u) \geq \alpha \|u\|^2_{1,D}. \quad \text{(7)}$$

Finally the family of numerical fluxes $((F_{K,\sigma}^D)_{K \in \mathcal{M}})_{\sigma \in \mathcal{E}}$ is said to be symmetric if for any $\mathcal{D} = (\mathcal{M}, \mathcal{E}, \mathcal{P}) \in \mathcal{F}$, the bilinear form defined by

$$\langle u, v \rangle_D = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{E}} F_{K,\sigma}^D(u)(v_K - v_\sigma), \quad \forall (u, v) \in X_{D,0}^2,$$

is such that

$$\langle u, v \rangle_D = \langle v, u \rangle_D, \quad \forall (u, v) \in X_{D,0}^2.$$

The finite volume scheme may then be written by approximating the integration of (1) in each control volume, and requiring that the scheme be conservative:

Find $u^D = ((u_K^D)_{K \in \mathcal{M}}, (u_\sigma^D)_{\sigma \in \mathcal{E}}) \in X_{D,0}$;

$$\sum_{\sigma \in \mathcal{E}_K} F_{K,\sigma}^D(u^D) = \int_K f(x) \, dx, \quad \forall K \in \mathcal{M}; \quad \text{(9)}$$

$$F_{K,\sigma}^D(u^D) + F_{L,\sigma}^D(u^D) = 0, \quad \forall \sigma \in \mathcal{E}_\text{int}, \mathcal{M}_\sigma = \{K, L\} \quad \text{(10)}$$

or, in equivalent form:

$$\text{Find } u^D = ((u_K^D)_{K \in \mathcal{M}}, (u_\sigma^D)_{\sigma \in \mathcal{E}}) \in X_{D,0} s.t. \langle u^D, v \rangle_D = \int_\Omega f(x) P_M v(x) \, dx, \quad \forall v \in X_{D,0}. \quad \text{(11)}$$

**Theorem 3.1.** Under assumptions (3), let $u$ be the unique solution to (4). Consider a regular family of admissible meshes $\mathcal{F}$, along with a family of consistent, coercive and symmetric fluxes $((F_{K,\sigma}^D)_{K \in \mathcal{M}})_{\sigma \in \mathcal{E}}$. Then, for all $\mathcal{D} \in \mathcal{F}$, there exists a unique $u^D \in X_{D,0}$ solution to (9) or (11), and $P_M u^D$ converges to $u$, solution of (4) in $L^q(\Omega)$, for all $q \in [1, +\infty]$ if $d = 2$ and all $q \in [1, 2d/(d - 2)]$ if $d > 2$, as $h_D \to 0$. Moreover, $\nabla u^D \in H_M(\Omega)^d$, defined by $m_K(\nabla u^D) = \sum_{\sigma \in \mathcal{E}_K} m_\sigma(u_\sigma - u_K)n_{K,\sigma}$ for all $K \in \mathcal{M}$, converges to $\nabla u$ in $L^2(\Omega)^d$.

**Sketch of proof.** Taking $v = u^D$ in (11), we get the following a priori estimate on $u^D$:

$$\alpha \|u^D\|^2_{1,D} \leq \|f\|_{L^2(\Omega)} \|u^D\|_{L^2(\Omega)}.$$

The discrete Sobolev inequality [1] holds thanks to the above definition of $\theta_D$, that is, there exists $C > 0$ depending only on $q, \Omega$ and $\theta$ such that: $\|P_M u^D\|_{L^q(\Omega)} \leq C \|P_M u^D\|_{1,M}$. Therefore, thanks to the fact that $\|P_M u^D\|_{1,M} \leq \|u^D\|_{1,D}$, we obtain that: $\|P_M u^D\|_{1,M} \leq \|u^D\|_{1,D} \leq \frac{C}{\alpha} \|f\|_{L^2(\Omega)}$, which yields the existence and uniqueness of $u^D$.

Then, prolonging by 0 the function $P_M u^D$ outside of $\Omega$, we get the estimate

$$\|P_M u^D(\cdot + \xi) - P_M u^D\|_{L^1(\mathbb{R}^d)} \leq \|\xi\| \left(\frac{1}{\alpha} \|m(\Omega)^{1/2}\|u^D\|_{1,D}, \quad \forall \xi \in \mathbb{R}^d.$$
We can therefore apply the Fréchet–Kolmogorov theorem, which is a compactness criterion in $L^1(\mathbb{R}^d)$. Again using the discrete Sobolev inequality, we get that, up to a subsequence, $P_M u^D$ converges, for all $q \in [1, +\infty)$ if $d = 2$ and all $q \in [1, 2d/(d-2))$ if $d > 2$, in $L^q(\mathbb{R}^d)$ to some function $\tilde{u}$, with $\tilde{u}(x) = 0$ for a.e. $x \in \mathbb{R}^d \setminus \Omega$. Furthermore, in the spirit of Lemma 2 of [4], we can show that $\nabla u^D$ converges to $\nabla \tilde{u}$ weakly in $L^2(\mathbb{R}^d)^d$. Therefore $\tilde{u} \in H^1_0(\Omega)$. To complete the proof of the theorem, we pass to the limit $h_D \to 0$ on the weak form of the scheme: for $\varphi \in C_0^\infty(\Omega)$, we take $v = P_D(\varphi)$ in (11). Using the symmetry and the consistency (6) of the fluxes $F^D_{K,\sigma}(\varphi)$, we obtain that (4) with $V$.

Therefore, by uniqueness, $\tilde{u} = u$ and the whole sequence converges. The strong convergence of $\nabla u^D$ to $\nabla u$ is obtained, using (7), the convergence of $\langle u^D, u^D \rangle_D$ to $\int_{\Omega} \nabla u \cdot \Lambda \nabla u \, dx$ and following the principles of the proof of Lemma 2.6 in [5].

4. An example of consistent, coercive and symmetric family of fluxes

Let us first note that the case of the classical four point finite volume schemes on triangles (also based on a consistent coercive and symmetric family of fluxes, see [6]) is included in the framework presented here. However, for general meshes or anisotropic diffusion operators, the construction of an approximation to the normal flux is more strenuous [2,3,7]; it is often performed by the reconstruction of a discrete gradient, either in the edges of the cell, or in the cell itself. We propose the following numerical fluxes, defined for $u \in X_D$ by

$$F_{K,\sigma}(u) = -m_{\sigma} \left( \nabla u_K \cdot \Lambda_K n_{K,\sigma} + \alpha_K \left( \frac{R_{K,\sigma}(u)}{d_{K,\sigma}} - \sum_{\sigma' \in E_K} m_{\sigma'} \frac{R_{K,\sigma'}(u)}{d_{K,\sigma'}} (x_{\sigma'} - x_K) \cdot \frac{n_{K,\sigma}}{m_K} \right) \right)$$

where $\Lambda_K$ is the mean value of the matrix $\Lambda(x)$ for $x \in K$, $\nabla u_K$ is defined in Theorem 3.1, $R_{K,\sigma}(u) = u_{\sigma} - u_K - \nabla u_K \cdot (x_\sigma - x_K)$, and $(\alpha_K)_{K \in M}$ is any family of strictly positive real numbers, bounded by above and below. We thus get a consistent, coercive and symmetric family of fluxes, in the above stated sense. In fact, in the same spirit as in the scheme derived in [5] for meshes satisfying an orthogonality condition, the above expression for $F_{K,\sigma}(u)$ is deduced from the variational form of the scheme, which is based on the following inner product:

$$\langle u, v \rangle_D = \sum_{K \in M} \left[ m_K \nabla u_K \cdot \Lambda_K \nabla v_K + \alpha_K \sum_{\sigma \in E_K} \frac{m_{\sigma}}{d_{K,\sigma}} R_{K,\sigma}(u) \tilde{R}_{K,\sigma}(v) \right], \quad \forall u, v \in X_D, 0.$$

References