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Probability Theory

On the set of solutions of a BSDE with continuous coefficient ☆

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Abstract

In this Note we prove that, if the coefficient $g = g(t, y, z)$ of a one-dimensional BSDE is assumed to be continuous and of linear growth in (y, z), then there exists either one or uncountably many solutions. ***To cite this article: G. Jia, S. Peng, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur l'ensemble des solutions d'une équation différentielle stochastique rétrograde avec coefficient continu. Nous prouvons dans cette Note que, si le coefficient $g = g(t, y, z)$ d'une EDSR est continu et linéairement croissant en (y, z), alors il existe soit une seule solution soit une infinité non dénombrable de solutions. ***Pour citer cet article : G. Jia, S. Peng, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

We consider the following one-dimensional backward stochastic differential equation:

$$y_t = \xi + \int_t^T g(s, y_s, z_s)\,\mathrm{d}s - \int_t^T z_s\,\mathrm{d}W_s, \quad t \in [0, T], \tag{1}$$

where the terminal condition ξ and the coefficient $g = g(t, y, z)$ are given. W is a d-dimensional Brownian motion. The solution $(y_t, z_t)_{t\in[0,T]}$ is a pair of square integrable processes. An interesting problem is: how many solutions does this BSDE have? In the standard situation where g satisfies linear growth condition and the Lipschitz condition in (y, z), it was proved by Pardoux and Peng [4] that there exists a unique solution. However in the case where g is only continuous in (y, z) without the Lipschitz restriction, Lepeltier and San Martin [3] have proved that there exists at least one solution. For more information about existence and uniqueness of BSDE, the reader can refer to [1] for discussions therein.

In this Note we will prove that if the coefficient g satisfies the conditions given in [3], then BSDE (1) has either one or uncountably many solutions. Our result also shows the structure of those solutions.

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2. The result and proof

Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(W_t)_{t\geqslant 0}$ be a d-dimensional standard Brownian motion on this space. Let $(\mathcal{F}_t)_{t\geqslant 0}$ be the filtration generated by this Brownian motion: $\mathcal{F}_t = \sigma\{W_s, s \in [0,t]\} \cup \mathcal{N}$, $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$, where $\mathcal{N}$ is the set of all P-null subsets.

Let $T > 0$ be a fixed real number. In this Note, we always work in the space $(\Omega, \mathcal{F}_T, P)$. For a positive integer n and $z \in \mathbb{R}^n$, we denote by $|z|$ the Euclidean norm of z.

We will denote by $\mathcal{H}_n^2 = \mathcal{H}_n^2(0,T;\mathbb{R}^n)$, the space of all $\mathbb{F}$-progressively measurable $\mathbb{R}^n$-valued processes such that $E[\int_0^T |\psi_t|^2\,\mathrm{d}t] < \infty$, and by $\mathcal{S}^2 = \mathcal{S}^2(0,T;\mathbb{R})$ the elements in $\mathcal{H}_n^2(0,T;\mathbb{R})$ with continuous paths such that

$$E\Big[\sup_{t\in[0,T]} |\psi_t|^2\Big] < \infty.$$

The coefficient g of BSDE is a function $g(\omega,t,y,z): \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$ satisfying the following assumptions:

(H1): linear growth: $\exists K < \infty$, s.t. $|g(\omega,t,y,z)| \leqslant K(1+|y|+|z|)$, $\forall t,\omega,y,z$,
(H2): $(g(t,y,z))_{t\in[0,T]} \in \mathcal{H}_1^2$, $\forall (y,z) \in \mathbb{R}\times\mathbb{R}^d$,
(H3): for fixed t,ω, $g(\omega,t,\cdot,\cdot)$ is continuous.

By Lepeltier and San Martin [3, Th. 1], under (H1)–(H3) and for each given $\xi \in L^2(\Omega,\mathcal{F}_T,P)$, there exists at least one solution $(y_t,z_t)_{t\in[0,T]} \in \mathcal{S}^2 \times \mathcal{H}_d^2$ of BSDE (1). [3] gives also the existence of the maximal solution $(\bar{y}_t,\bar{z}_t)_{t\in[0,T]}$ and the minimal solution $(\underline{y}_t,\underline{z}_t)_{t\in[0,T]}$ of BSDE (1) in the sense that any solution $(y_t,z_t)_{t\in[0,T]} \in \mathcal{S}^2\times\mathcal{H}_d^2$ of BSDE (1) must satisfy $\underline{y}_t \leqslant y_t \leqslant \bar{y}_t$, a.s., for all $t\in[0,T]$.

Theorem 2.1. *We assume* (H1)–(H3). *Let* $(\underline{y}_t,\underline{z}_t)_{t\in[0,T]} \in \mathcal{S}^2\times\mathcal{H}_d^2$ *and* $(\bar{y}_t,\bar{z}_t)_{t\in[0,T]} \in \mathcal{S}^2\times\mathcal{H}_d^2$ *be the minimal and maximal solution of BSDE* (1) *with the terminal condition* $\xi \in L^2(\Omega,\mathcal{F}_T,P)$. *Then for each* $t_0\in[0,T]$ *and* $\eta \in L^2(\Omega,\mathcal{F}_{t_0},P)$ *such that*

$$\underline{y}_{t_0} \leqslant \eta \leqslant \bar{y}_{t_0}, \quad a.s.,$$

there exists at least one solution $(y_t,z_t)_{t\in[0,T]} \in \mathcal{S}^2\times\mathcal{H}_d^2$ *of BSDE* (1) *satisfying*

$$y_{t_0} = \eta, \quad a.s.$$

Proof. Let $(y_t^1,z_t^1)_{t\in[0,t_0]} \in \mathcal{S}^2(0,t_0)\times\mathcal{H}_d^2(0,t_0;\mathbb{R}^d)$ be a solution of the following BSDE

$$y_t^1 = \eta + \int_t^{t_0} g\big(s,y_s^1,z_s^1\big)\,\mathrm{d}s - \int_t^T z_s^1\,\mathrm{d}W_s, \quad t\in[0,t_0]$$

and, for a fixed $z^2 \in \mathcal{H}_d^2(t_0,T;\mathbb{R}^d)$, let $(y_t^2)_{t\in[t_0,T]}$ be a (strong) solution of the SDE

$$y_t^2 = \eta - \int_{t_0}^t g\big(s,y_s^2,z_s^2\big)\,\mathrm{d}s + \int_{t_0}^t z_s^2\,\mathrm{d}W_s, \quad t\in[t_0,T].$$

We define a stopping time $\tau = \inf\{t \geqslant t_0,\ y_t^2 \notin (\underline{y}_t,\bar{y}_t)\}$. By $\underline{y}_T = \bar{y}_T$, we know that $\tau \leqslant T$. Now we define on $[0,T]$

$$(y_t,z_t) = \mathbb{I}_{[0,t_0)}(t)\big(y_t^1,z_t^1\big) + \mathbb{I}_{[t_0,\tau)}(t)\big(y_t^2,z_t^2\big) + \mathbb{I}_{[\tau,T]}(t)(\bar{y}_t,\bar{z}_t)\mathbb{I}_{\{y_\tau=\bar{y}_\tau\}} + \mathbb{I}_{[\tau,T]}(t)(\underline{y}_t,\underline{z}_t)\mathbb{I}_{\{y_\tau<\bar{y}_\tau\}}.$$

One can easily check that $(y_t,z_t)_{t\in[0,T]} \in \mathcal{S}^2(0,T)\times\mathcal{H}_d^2(0,T;\mathbb{R}^d)$ and is a solution of BSDE (1) with $y_T = \xi$ and $y_{t_0} = \eta$. □

Remark 1. The linear growth assumption (H1) can be replaced by a quadratic growth assumption of type given in [2].

Remark 2. Indeed, when BSDE (1) has uncountably many solutions, the cardinality of the associated solution set is at least continuum since we can take $\eta = \alpha \underline{y}_{t_0} + (1-\alpha)\bar{y}_{t_0}$ for each $\alpha \in [0, 1]$.

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