Probability Theory

On the set of solutions of a BSDE with continuous coefficient

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Abstract

In this Note we prove that, if the coefficient \( g = g(t, y, z) \) of a one-dimensional BSDE is assumed to be continuous and of linear growth in \((y, z)\), then there exists either one or uncountably many solutions. To cite this article: G. Jia, S. Peng, C. R. Acad. Sci. Paris, Ser. I 344 (2007).

Résumé

Sur l’ensemble des solutions d’une équation différentielle stochastique rétrograde avec coefficient continu. Nous prouvons dans cette Note que, si le coefficient \( g = g(t, y, z) \) d’une EDSR est continu et linéairement croissant en \((y, z)\), alors il existe soit une seule solution soit une infinité non dénombrable de solutions. Pour citer cet article : G. Jia, S. Peng, C. R. Acad. Sci. Paris, Ser. I 344 (2007).

1. Introduction

We consider the following one-dimensional backward stochastic differential equation:

\[
y_t = \xi + \int_t^T g(s, y_s, z_s) \, ds - \int_t^T z_s \, dW_s, \quad t \in [0, T],
\]

where the terminal condition \( \xi \) and the coefficient \( g = g(t, y, z) \) are given. \( W \) is a \( d \)-dimensional Brownian motion. The solution \((y_t, z_t)_{t \in [0, T]}\) is a pair of square integrable processes. An interesting problem is: how many solutions does this BSDE have? In the standard situation where \( g \) satisfies linear growth condition and the Lipschitz condition in \((y, z)\), it was proved by Pardoux and Peng [4] that there exists a unique solution. However in the case where \( g \) is only continuous in \((y, z)\) without the Lipschitz restriction, Lepeltier and San Martin [3] have proved that there exists at least one solution. For more information about existence and uniqueness of BSDE, the reader can refer to [1] for discussions therein.

In this Note we will prove that if the coefficient \( g \) satisfies the conditions given in [3], then BSDE (1) has either one or uncountably many solutions. Our result also shows the structure of those solutions.

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2. The result and proof

Let \((\Omega, \mathcal{F}, P)\) be a probability space and \((W_t)_{t \geq 0}\) be a \(d\)-dimensional standard Brownian motion on this space. Let \((\mathcal{F}_t)_{t \geq 0}\) be the filtration generated by this Brownian motion: \(\mathcal{F}_t = \sigma\{W_s, s \in [0, t]\} \cup \mathcal{N}\), \(\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}\), where \(\mathcal{N}\) is the set of all \(P\)-null subsets.

Let \(T > 0\) be a fixed real number. In this Note, we always work in the space \((\Omega, \mathcal{F}, P)\). For a positive integer \(n\) and \(z \in \mathbb{R}^n\), we denote by \(|z|\) the Euclidean norm of \(z\).

We will denote by \(\mathcal{H}_n^2 = \mathcal{H}_n^2(0; \mathbb{R}^d)\), the space of all \(\mathbb{F}\)-progressively measurable \(\mathbb{R}^n\)-valued processes such that \(E[\int_0^T |\psi_t|^2 \, dt] < \infty\), and by \(\mathcal{S}_2^2 = \mathcal{S}_2^2(0; \mathbb{R})\) the elements in \(\mathcal{H}_2^2(0, T; \mathbb{R})\) with continuous paths such that

\[
E\left[\sup_{t \in [0, T]} |\psi_t|^2\right] < \infty.
\]

The coefficient \(g\) of BSDE is a function \(g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}\) satisfying the following assumptions:

(H1): linear growth: \(\exists K < \infty, \text{s.t. } |g(\omega, t, y, z)| \leq K(1 + |y| + |z|), \forall t, \omega, y, z,\)

(H2): \((g(t, y, z))_{t \in [0, T]} \in \mathcal{H}_1^2, \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d,\)

(H3): for fixed \(t, \omega, g(\omega, t, \cdot, \cdot)\) is continuous.

By Lepeltier and San Martin [3, Th. 1], under (H1)–(H3) and for each given \(\xi \in L^2(\Omega, \mathcal{F}, P)\), there exists at least one solution \((y_t, z_t)_{t \in [0, T]} \in \mathcal{S}_2^2 \times \mathcal{H}_d^2\) of BSDE (1). [3] gives also the existence of the maximal solution \((\bar{y}_t, \bar{z}_t)_{t \in [0, T]}\) and the minimal solution \((\underline{y}_t, \underline{z}_t)_{t \in [0, T]}\) of BSDE (1) in the sense that any solution \((y_t, z_t)_{t \in [0, T]} \in \mathcal{S}_2^2 \times \mathcal{H}_d^2\) of BSDE (1) must satisfy \(\underline{y}_t \leq y_t \leq \bar{y}_t\), a.s., for all \(t \in [0, T]\).

**Theorem 2.1.** We assume (H1)–(H3). Let \((\underline{y}_t, \underline{z}_t)_{t \in [0, T]} \in \mathcal{S}_2^2 \times \mathcal{H}_d^2\) and \((\bar{y}_t, \bar{z}_t)_{t \in [0, T]} \in \mathcal{S}_2^2 \times \mathcal{H}_d^2\) be the minimal and maximal solution of BSDE (1) with the terminal condition \(\xi \in L^2(\Omega, \mathcal{F}, P)\). Then for each \(t_0 \in [0, T]\) and \(\eta \in L^2(\Omega, \mathcal{F}_{t_0}, P)\) such that

\[
y_{t_0} \leq \eta \leq \bar{y}_{t_0}, \quad \text{a.s.},
\]

there exists at least one solution \((y_t, z_t)_{t \in [0, T]} \in \mathcal{S}_2^2 \times \mathcal{H}_d^2\) of BSDE (1) satisfying

\[
y_{t_0} = \eta, \quad \text{a.s.}
\]

**Proof.** Let \((y_t^1, z_t^1)_{t \in [0, t_0]} \in \mathcal{S}_2^2(0, t_0) \times \mathcal{H}_d^2(0, t_0; \mathbb{R}^d)\) be a solution of the following BSDE

\[
y_t^1 = \eta + \int_0^t g(s, y_s^1, z_s^1) \, ds - \int_0^t z_s^1 \, dW_s, \quad t \in [0, t_0]
\]

and, for a fixed \(z_\tau \in \mathcal{H}_d^2(t_0, \tau; \mathbb{R}^d)\), let \((y_\tau^2)_{t \in [t_0, \tau]}\) be a (strong) solution of the SDE

\[
y_t^2 = \eta - \int_{t_0}^t g(s, y_s^2, z_s^2) \, ds + \int_{t_0}^t z_s^2 \, dW_s, \quad t \in [t_0, \tau].
\]

We define a stopping time \(\tau = \inf\{t \geq t_0, y_t^2 \notin (\underline{y}_t, \bar{y}_t)\}\). By \(y_{t_0} = \bar{y}_{t_0}\), we know that \(\tau \leq T\). Now we define on \([0, T]\)

\[
(y_t, z_t) = \begin{cases} (y_0, t)(y_t^1, z_t^1) + I_{t_0, \tau}(t)(y_t^2, z_t^2) + I_{\tau, T}(t)(\bar{y}_t, \bar{z}_t)I_{\bar{y}_t = \bar{y}_1} + I_{\tau, T}(t)(\underline{y}_t, \underline{z}_t)I_{\bar{y}_t < \bar{y}_1}, \end{cases}
\]

One can easily check that \((y_t, z_t)_{t \in [0, T]} \in \mathcal{S}_2^2(0, T) \times \mathcal{H}_d^2(0, T; \mathbb{R}^d)\) and is a solution of BSDE (1) with \(y_T = \xi\) and \(y_{t_0} = \eta\).

**Remark 1.** The linear growth assumption (H1) can be replaced by a quadratic growth assumption of type given in [2].
Remark 2. Indeed, when BSDE (1) has uncountably many solutions, the cardinality of the associated solution set is at least continuum since we can take $\eta = \alpha y_{\leq t_0} + (1 - \alpha)\tilde{y}_{t_0}$ for each $\alpha \in [0, 1]$.

References