Differential Geometry

Elliptic genera of level $N$ on complex $\pi_2$-finite manifolds

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Received 27 June 2006; accepted after revision 22 January 2007

Abstract

We prove the rigidity of the elliptic genera of level $N$ on complex manifolds with finite second homotopy group admitting circle actions, and the vanishing of the Hilbert polynomial of its canonical bundle. To cite this article: R. Herrera, C. R. Acad. Sci. Paris, Ser. I 344 (2007).

1. Introduction

The elliptic genus was introduced by Ochanine [6] and re-interpreted by Witten [8], who conjectured its rigidity under circle actions on spin manifolds. The rigidity of the elliptic genus was proved by Taubes [7], Bott and Taubes [1], etc., and was generalized to non-spin manifolds with finite second homotopy group in [3]. Furthermore, Witten and Hirzebruch proposed independently a complex version of the genus in the form of the elliptic genus of level $N > 0$ for complex manifolds with $c_1 \equiv 0 \pmod{N}$ and conjectured its rigidity [9], which was proved by Hirzebruch [4], Krichever [5], etc. In this note, we prove the rigidity of the elliptic genus of level $N$ on $\pi_2$-finite complex manifolds $M$ for all $N > 0$ (see Theorem 2.1), which in turn implies the vanishing of the Hilbert polynomial $\chi(M, K^k) = 0$ for all $k$, where $K$ denotes the canonical bundle of $M$ (see Corollary 3.1).

The note is organized as follows: in Section 2 we give the definition of the elliptic genus of level $N$ and state the Rigidity Theorem 2.1, and in Section 3 we sketch its proof.

1 Partially supported by a JSPS Research Fellowship PE-05030, PSC-CUNY award #67300-00-36, Convenio CONCYTEG 05-02-K117-112, and Apoyo CONACYT J48320-F.

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2. Rigidity of the elliptic genera of level \(N\)

Let \(M\) be a \(d\)-dimensional compact manifold and \(T\) its holomorphic tangent bundle. The elliptic genus of level \(N\) defined by Witten has the following \(q\)-development in the standard cusp of \(\Gamma_1(N) \subset SL_2(\mathbb{Z})\), which we shall take as its definition

\[
\tilde{\phi}_N(M) = \sum_{j=0}^{\infty} \chi_y(M, R_j)q^j,
\]

where \(-y = \zeta = e^{2\pi i/N}\), and the \(R_j\) denote virtual vector bundles with coefficients in \(\mathbb{Z}[\zeta]\) arising from the following infinite product

\[
R(q, T) = \sum_{j=0}^{\infty} R_j q^j = \bigotimes_{j=1}^{\infty} yq^j \bigotimes_{j=1}^{\infty} S q^j (T + T^*),
\]

where

\[
\bigotimes_t(W) = \sum_{j=0}^{\text{rk}(W)} \bigotimes_j W \cdot t^j \quad \text{and} \quad S_t(W) = \sum_{j=0}^{\infty} S^j W \cdot t^j
\]

denote the sums of exterior and symmetric powers of a vector bundle \(W\), respectively. The first two terms are

\[
R_0 = 1, \quad R_1 = (1 - \zeta)T^* + (1 - \zeta^{-1})T.
\]

Thus we can see that this \(q\)-development has integral coefficients. The first term of \(\tilde{\phi}(M)\) is \(\chi_y(M)\).

The \(q\)-developments at other cusps, however, have coefficients which are not necessarily integral. Such coefficients are of the form \(\chi(M, K k/N \otimes W_n)\) for some virtual vector bundle \(W_n\), and the non-integrality may happen due to \(K\) not necessarily admitting an \(N\)-th root. For instance, the first term of the expansion at a cusp of the form \(2\pi i k\tau/N\) for \(1 \leq k \leq N\) is

\[
\frac{1}{\tilde{q}^{k(N-k)d/2N}} \chi(M, K^{k/N}),
\]

where \(\tilde{q}\) is a uniformizing parameter for this cusp (\(\tilde{q}^N = q\)).

If we assume that \(M\) admits a holomorphic \(S^1\)-action, there is an induced action on the bundles \(R_j\) and on the cohomology groups \(H^p(M, \bigotimes^p T^* \otimes R_j)\). Thus, the traces of such action on the cohomology groups produce the \(S^1\)-character \(\chi(M, \bigotimes^p T^* \otimes R_j, \lambda)\), where \(\lambda \in S^1\), so that

\[
\chi_y(M, R_j, \lambda) = \sum_{p=0}^{d} \chi(M, \bigotimes^p T^* \otimes R_j, \lambda) y^p, \quad \tilde{\phi}(M, \lambda) = \sum_{j=0}^{\infty} \chi_y(M, R_j, \lambda)q^j.
\]

The rigidity of the elliptic genus for the \(S^1\)-action means that the finite Laurent series \(\chi(M, R_j, \lambda)\) does not depend on \(\lambda\) and, therefore, \(\tilde{\phi}(M, \lambda)\) is constant in that variable. Thus, we can now state the rigidity theorem:

**Theorem 2.1.** Let \(M\) be a compact complex manifold with finite second homotopy group, and admitting a non-trivial holomorphic \(S^1\)-action. Then, the equivariant elliptic genus \(\tilde{\phi}_N(M, \lambda)\) does not depend on \(\lambda \in S^1\), i.e.

\[
\tilde{\phi}_N(M, \lambda) = \tilde{\phi}_N(M).
\]

3. Sketch of proof

Hirzebruch’s proof of the rigidity theorem [4] for the elliptic genus of level \(N\) considers a normalized version, applies the Atiyah–Bott–Singer fixed point theorem (holomorphic Lefschetz theorem) and examines the behaviour of the resulting meromorphic expressions. The normalized elliptic genus is

\[
\varphi(M, \lambda) = \frac{\tilde{\phi}(M, \lambda)}{Y(-\alpha)d} = \sum_{j=0}^{\infty} \chi_y(M, S_j, \lambda)q^j.
\]
where \( S_j \) are virtual vector bundles with coefficients in \( \mathbb{Q}(\zeta) \),
\[
Y(x) = (1 - e^{-x}) \prod_{j=1}^{\infty} \frac{(1 - q^j e^{-x})(1 - q^j e^x)}{1 - q^j},
\]
\( \alpha = 2\pi i / N \). By applying the Atiyah–Singer–Bott fixed point theorem
\[
\varphi(M, \lambda) = \sum_{v} \varphi_N(M, \lambda)_v,
\]
where \( v \) is an index for the connected components \( M^S_1 \) of \( M^S \),
\[
\varphi_N(M, \lambda)_v = \langle e_0 \cdot F(x_1 + 2\pi im_1 z) \cdots F(x_d + 2\pi im_d z), [M^S_1] \rangle
\]
e\( 0 \) is the Euler class of \( M^S_1 \), \( F(x) = Y(x - \alpha)/(Y(x)Y(-\alpha)) \), the \( m_i \) are the exponents of the infinitesimal action of \( S^1 \) on \( T|_{M^S_1} = L^{m_1} \oplus \cdots \oplus L^{m_d} \), and \( x_i \) is the formal root of each one of the lines into which \( T \) splits.

If the first Chern class of \( M \) is divisible by \( N \) then

(i) \( \varphi_N(M, \lambda) \) is elliptic with respect to a certain lattice;
(ii) \( \varphi_N(M, \lambda) \) has no poles, which implies it is holomorphic and, therefore, constant in \( \lambda \).

For (i), what is really needed is the \( S^1 \)-action to be \( N \)-balanced. A circle action is called \( N \)-balanced if the residue class of the sum
\[
m_1 + \cdots + m_d \pmod{N}
\]
does not depend on the connected component \( M^S_1 \). The common residue is called the type \( t \) of the \( S^1 \)-action.

**Theorem 3.1.** [4, p. 179] For an \( N \)-balanced \( S^1 \)-action of type \( t \) on the complex manifold \( M \), the equivariant elliptic genus \( \varphi_N(M, \lambda) \), with \( \lambda = e^{2\pi i z} \), is an elliptic function for the lattice \( \mathbb{Z} \cdot N \tau + \mathbb{Z} \) which satisfies
\[
\varphi_N(M, \lambda q) = \zeta^t \varphi_N(M, \lambda), \quad (\zeta = e^{2\pi i/N}).
\]

For (ii), we have to consider the sums
\[
\psi(\lambda) = \sum_{M^S_1 \subset X} \varphi_N(M, \lambda)_v
\]
for those \( M^S_1 \) contained in a given connected component \( X \) of the fixed point set \( M^{Z_m} \), \( Z_m \subset S^1 \), for every \( m \in \mathbb{Z} \). Hirzebruch determined that \( \psi(\lambda, q^{s/m}) \), for any integer \( s \), has no poles on the unit circle as long as the residues
\[
\sum_{i=1}^{d} \left[ \frac{m_i}{m} \right] \pmod{mN}
\]
are all equal. In this way, \( \varphi_N(M, \lambda) \) has no poles at all and the rigidity theorem follows if \( c_1(M) \equiv 0 \pmod{N} \)
\[
\varphi_N(M, \lambda)_v = \varphi_N(M).
\]

However, conditions (i) and (ii) on the \( S^1 \)-action are also fulfilled by actions on complex manifolds with finite second homotopy group. Consider the \( S^1 \)-decompositions of the tangent space at two distinct \( S^1 \)-fixed points \( p \) and \( p' \) in terms of generator \( L \cong \mathbb{C} \) of the representation ring \( R(S^1) \)
\[
T_p M = L^{m_1} \oplus \cdots \oplus L^{m_d}, \quad T_{p'} M = L^{m'_1} \oplus \cdots \oplus L^{m'_d},
\]
where \( m_i \) and \( m'_i \) are the exponents of the \( S^1 \) action at \( p \) and \( p' \), respectively. By [2], the virtual representation \( T_p - T_{p'} \) can be factored as follows
\[
T_p - T_{p'} = (1 - L)^2 \otimes \left( \bigoplus_j b_j L^j \right).
\]
where the set \( \{ b_j \in \mathbb{Z} \} \) is finite. Thus
\[
\sum_{i=1}^{d} m_i - \sum_{i=1}^{d} m'_i = \sum_j b_j \cdot j - 2 \sum_j b_j \cdot (j + 1) + \sum_j b_j \cdot (j + 2) = 0.
\]
Hence, conditions (i) and (ii) hold, and the Rigidity Theorem follows.

As a consequence, we see that
\[
\varphi_N(M) = \varphi_N(M, \lambda q) = \zeta^t \varphi_N(M, \lambda) = \zeta^t \varphi_N(M),
\]
so that, if the \( S^1 \)-action has type \( t \neq 0 \) then
\[
\varphi_N(M) \equiv 0 \quad \text{and} \quad \tilde{\varphi}_N(M) \equiv 0.
\]

On the other hand, the Rigidity Theorem readily implies that the not necessarily integral characteristic numbers
\[
\chi(M, K^k) = 0
\]
for \( k = 1, \ldots, N - 1 \), as in [4]. Since we have imposed no divisibility condition on the first Chern class of \( M \), these vanishings hold for any \( N \). Thus, the Hilbert polynomial \( \chi(M, K^k) \) has infinitely many zeroes and is, therefore, identically zero.

**Corollary 3.1.** Let \( M \) be a compact complex manifold with finite second homotopy group, and admitting a non-trivial holomorphic \( S^1 \)-action. Then
\[
\chi(M, K^k) = 0 \quad \text{for all } k.
\]
In particular, the Todd genus vanishes, \( \text{Todd}(M) = 0 \).

Hence, the Todd genus is an obstruction to the existence of holomorphic circle actions on \( \pi_2 \)-finite compact complex manifolds.

**Acknowledgement**

The author wishes to thank the Max Planck Institute for Mathematics (Bonn) for its hospitality and support.

**References**