Numerical Analysis

The Khokhlov–Zabolotskaya–Kuznetsov equation

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Abstract

For the KZK equation \((u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0\) in the class of \(x\)-periodic and of zero mean value functions we have analysed the following: the derivation from Navier–Stokes system and the validity of its approximation, the existence, uniqueness and stability of the solution. The solution is proved to be global in time for sufficient small initial data with \(\beta > 0\) and to have a blow-up if \(\beta = 0\).

Résumé

Équation de Khokhlov–Zabolotskaya–Kuznetsov. Pour l’équation KZK \((u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0\) dans la classe des fonctions périodiques en \(x\) et de moyennes nulles, on a étudié la dérivation à partir du système de Navier–Stokes isentropique et la validation de son approximation, l’existence, l’unicité et la stabilité de la solution. On a prouvé que la solution est globale en temps pour des données initiales suffisamment petites avec \(\beta > 0\) et que la solution présente une onde de choc si \(\beta = 0\).

Version française abrégée

On considère l’équation

\[(u_t - uu_x - \beta u_{xx})_x - \gamma \Delta_y u = 0, \ \ (x, y) \in \mathcal{R}_x/(LZ) \times \mathcal{R}_y^{n-1}.\] (1)

Elle porte le nom de KZK (Khokhlov–Zabolotskaya–Kuznetsov) et a été introduite pour étudier des problèmes d’acoustique non linéaire [2] comme approximation d’une équation de Navier–Stokes isentropique

\[\partial_t \rho_\epsilon + \nabla (\rho_\epsilon u_\epsilon) = 0, \ \ \rho[\partial_t u_\epsilon + (u_\epsilon \cdot \nabla)u_\epsilon] = -\nabla p(\rho_\epsilon) + \epsilon \nu \Delta u_\epsilon,\] (2)

avec équation d’état approchée \(\rho_\epsilon = \rho_0 + \epsilon \bar{\rho}_\epsilon,\)

\[p = p(\rho_\epsilon) = c^2 \epsilon \bar{\rho}_\epsilon + \frac{(\gamma - 1)c^2}{2\rho_0} \epsilon^2 \bar{\rho}_\epsilon^2, \ \ et \ c = \sqrt{p'(\rho_0)} \ la \ vitesse \ du \ son.\] (3)
Il s’agit de décrire des solutions qui se propagent comme des faisceaux (comme dans l’approximation paraxiale), mais dans un cadre non linéaire (les effets non linéaires et la viscosité sont du même ordre que les oscillations longitudinales).1


1. Existence, stability, uniqueness and blow-up results

We consider the KZK equation (1) involving two positive constants \( \beta \) and \( \gamma \) with the condition of periodicity and of mean value zero (which corresponds to practical situations [2])

\[
\begin{align*}
  u(x + L, y, t) &= u(x, y, t), \quad \int_{0}^{L} u(x, y, t) \, dx = 0.
\end{align*}
\]

We define the inverse of the derivative \( \partial_x^{-1} \) as an operator acting in the space of periodic functions with mean value zero

\[
\partial_x^{-1} f = \int_{0}^{x} f(s) \, ds + \int_{0}^{L} \frac{s}{L} f(s) \, ds.
\]

Therefore (1) is equivalent to the equation

\[
\begin{align*}
  u_t - uu_x - \beta u_{xx} - \gamma \partial_x^{-1} \Delta_y u &= 0 \quad \text{in} \quad \mathcal{R}_x/(L \mathbb{Z}) \times \Omega.
\end{align*}
\]

Finally when \( \gamma = 0 \) (1) reduces to the Burgers–Hopf equation for which existence smoothness and uniqueness of solution are well-known. For \( \gamma = \beta = 0 \) it reduces to the Burgers equation which after a finite time exhibits singularities.2

To prove the existence theorem we use a priori estimates for smooth solutions of the integrated KZK equation (6) (the \( L^2 \) norm and the \( H^s \) in \( \mathbb{R}^x/(L \mathbb{Z}) \times \mathbb{R}^n \)) are denoted by \( |u| \) and by \( \|u\|_s \)

\[
\begin{align*}
  \frac{1}{2} \frac{d}{dt} |u(\cdot, \cdot, t)|^2 + \beta |\partial_x u(\cdot, \cdot, t)|^2 &= 0, \\
  \text{for} \quad s > \left[ \frac{n}{2} \right] + 1 \\
  &\quad \frac{1}{2} \frac{d}{dt} \|u\|^2_s + \beta \|\partial_x u\|^2_s \leq C(s) \|u\|^3_s
\end{align*}
\]

and

\[
\begin{align*}
  \frac{1}{2} \frac{d}{dt} \|u\|^2_s + \beta C(L) \|u\|^2_s \leq C(s) \|u\|^3_s.
\end{align*}
\]

The estimates (8), (9) are valid for \( s > \left[ \frac{n}{2} \right] + 1 \) which is necessary for the use of the Sobolev theorem. The estimate (7) shows that we have the conservation of \( L^2 \)-norm if \( \beta = 0 \). We obtain the following:

**Theorem 1.1.** For the Cauchy problem

\[
\begin{align*}
  u_t - uu_x - \beta u_{xx} - \gamma \partial_x^{-1} (\Delta_y u) &= 0, \quad u(x, y, 0) = u_0
\end{align*}
\]

considered in \( \mathcal{R}_x/(L \mathbb{Z}) \times \mathcal{R}^n_y \) with (4), and with \( \beta \geq 0 \) one has the following results:

(i) For \( s > \left[ \frac{n}{2} \right] + 1 \) there exists a constant \( C(s, L) \) such that for any initial data \( u_0 \in H^s \) the problem (10) has on an interval \( [0, T] \) with \( T \geq \frac{1}{C(s, L)\|u_0\|_{H^s}} \) a solution in \( C([0, T], H^s) \cap C^1([0, T], H^{s-2}) \).
(ii) Let \( T^* \) be the biggest time on which such solution is defined then one has
\[
\int_0^{T^*} \sup_{x,y} \left( |\partial_x u(x, y, t)| + |\nabla_y u(x, y, t)| \right) dt = \infty.
\] (11)

(iii) If \( \beta > 0 \) there exists a constant \( C_1 \) such that \( \| u_0 \|_s \leq C_1 \Rightarrow T^* = \infty \).

(iv) For two solutions \( u \) and \( v \) of KZK equation, with \( u \in L_\infty([0, T[; H^s) \) and \( v \in L^2([0, T[; L_2) \), one has the following stability uniqueness result:
\[
|u(\cdot, t) - v(\cdot, t)|_{L^2} \leq e^{\int_0^t \sup_{x,y} |\partial_x u(x, y, s)| ds} |u(\cdot, 0) - v(\cdot, 0)|_{L^2}.
\] (12)

**Remark 1.** The estimate (12) is of strong-weak form: as in [3] only the \( L_\infty \) norm of \( u_x \) is needed.

**Remark 2.** When there is no viscosity all the corresponding statements of Theorem 1.1 remain valid for \( 0 > t > -C \) with a convenient \( C \).

**Theorem 1.2.** The equation
\[
(ut - uux)_x - \gamma \Delta_y u = 0 \quad \text{in } \mathcal{R}_{t+} \times \mathcal{R}_c \times \Omega
\] (13)
with Neumann boundary condition on \( \partial \Omega \) has no global in time smooth solution if \( \sup_{x,y} \partial_x u(x, y, 0) \) is large enough with respect to \( \gamma \).

**Remark 3.** As we can see from [2] the result of the theorem confirms the numerical results. One observes that for \( \beta \to 0 \) the solution of the KZK equation has a quasi-shock approaching the shock wave, into which it degenerates for \( \beta = 0 \).

The proof of the blow-up follows the ideas of S. Alinhac [1]. First the blow-up is observed for \( \gamma = 0 \) and related to a singularity in the projection of an unfolded ‘blow-up system’. Second the properties of this unfolded blow-up system are shown to be stable under small perturbations. One uses a Nash–Moser theorem with tamed estimates and this is the reason why will exists a \( T^* \) such that:
\[
\lim_{t \to T^*} (T^* - t) \sup_{x,y} \partial_x u(x, y, t) > 0.
\]

2. Approximation results

One finds in physical works [2,8] a formal derivation of the KZK equation as a second order approximation of the isentropic Navier–Stokes system (2) with the approximate state equation (3). We offer here to give a strict proof and orders of magnitude of the approximation.

We study solutions propagating along the axis \( x_1 \) direction. Therefore it is assumed that its variation in the direction \( x' = (x_2, x_3, \ldots, x_n) \) perpendicular to the \( x_1 \) axis is much larger that its variation along the axis \( x_1 \). We use the ansatz
\[
\rho_e = \rho_0 + \epsilon \tilde{\rho}_e = \rho_0 + \epsilon I \left( t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x' \right) = \rho_0 + \epsilon I(\tau, z, y),
\]
\[
u_e = \epsilon \tilde{v}_e = \epsilon (\underline{u}_e, \underline{u}_e') = \epsilon (v + \epsilon v_1, \sqrt{\epsilon} \tilde{w})(t - \frac{x_1}{c}, \epsilon x_1, \sqrt{\epsilon} x'),
\]
\[
u_e = \epsilon (v + \epsilon v_1, \sqrt{\epsilon} \tilde{w})(\tau, z, y),
\]
here \( \epsilon \) is a dimensionless parameter which characterizes the smallness of the perturbation. For instance in water with a initial power of the order of 0.3 \( \text{Vt/cm}^2 \) \( \epsilon = 10^{-5} \). We obtain
\[
\epsilon (\rho_0 v - c I) = 0 \Rightarrow v(\tau, z, y) = \frac{c}{\rho_0} I(\tau, z, y),
\]
\[
\epsilon^{3/2} (\rho_0 \partial_t \tilde{w} + c^2 \nabla y I) = 0 \Rightarrow \tilde{w}(\tau, z, y) = \frac{c^2}{\rho_0} \left( \int_0^\tau \nabla y I(s, z, y) ds + \int_0^L \frac{s}{L} \nabla y I(s, z, y) ds \right).
\]
Theorem 2.1. We take the initial data for the KZK equation with (cf. [4,6] for the related questions for Navier–Stokes system in the half space).

We consider the exact system (2) denoted \( \Phi(\rho_t, u_t, v) = 0 \) and the KZK solution \( I(\tau, z, y) = I(t - \frac{\tau}{\epsilon}, \epsilon x_1, \sqrt{\epsilon} x') \) of (19).

The viscosity \( \nu \) introduces some difference in the construction.

With no viscosity both the nonlinear system of elasticity [3] and the KZK equation are well posed for positive and negative but finite time, so, and since \( z = \epsilon x_1 \) (as soon as \( z \) becomes the time variable according to the KZK derivation), the problem of approximation is considered in the half space

\[
\frac{\partial}{\partial \tau} I = \frac{\nu}{2\rho_0} c_\tau^2 \Delta y I + \frac{c(\nu - 1)}{4L\rho_0^2} \int_0^L I^2 d\tau + \frac{3\nu}{2c\rho_0} \partial_t I.
\]

\( \nu \) is such that for any \( \epsilon \) small enough the solutions \( U_\epsilon \) of the exact system and \( \tilde{U}_\epsilon \) of the approximate system \( \Phi(\tilde{U}_\epsilon, 0) = \epsilon^{5/2} R \) (the functions \( \tilde{u}_\epsilon, \tilde{\rho}_\epsilon \) are constructed according to formulas (14), (15) and the rest \( R \) is bounded), which have been determinate as above in the cone \( C(T) \) with the same initial data (21), satisfy the estimate

\[
\| \nabla : U_\epsilon \|_{L^\infty(0, T^\epsilon_0; H^{n-1})} < \epsilon C \quad \text{for} \ s > \left[ \frac{n}{2} \right] + 1.
\]

Then there exists a constant \( C \) such that for any \( \epsilon \) small enough the solutions \( U_\epsilon \) of the exact system and \( \tilde{U}_\epsilon \) of the approximate system \( \Phi(\tilde{U}_\epsilon, 0) = \epsilon^{5/2} R \) are bounded, which have been determinate as above in the cone \( C(T) \) with the same initial data (21), satisfy the estimate

\[
\| \tilde{U}_\epsilon - U_\epsilon \|_{L^2(Q_\epsilon(t))} \leq \epsilon^{5} C \| \nabla \tilde{U}_\epsilon \|_{L^\infty(C(T)\, t)} \leq \epsilon^{5} C t.
\]

The estimate (22) also holds in the norm of \( H^{n'} \) for \( s' = s - 4 \) as soon as in the cone the booth solutions \( \tilde{U}_\epsilon \) and \( U_\epsilon \) are in \( C([0, T^\epsilon_0]; H^{n'}) \).

For the viscous case we consider the solution of the KZK equation for \( \nu > 0 \) and the correctors \( v, w \) and \( v_1 \) given by (16)–(18). First we observe

Proposition 2.2. Suppose that the initial data of the KZK Cauchy problem \( I_0(t, y) = I_0(t, \sqrt{\epsilon} x') \) is such that

(i) it is periodic on \( t \) with the period \( L \) and of mean value zero,

(ii) for fixed \( t \) it has the same sign for all \( y \in \mathbb{R}^{n-1} \), and for \( t \in ]0, L[ \) change the sign, i.e., \( I_0 = 0 \), only finite number times,
(iii) \( I_0(t, y) \in H^s(t \geq 0) \times \mathcal{R}^{n-1} \) for \( s' > \max\{6, \frac{n}{2}+1\} \).

(iv) \( I_0 \) is sufficiently small in the sense of Theorem 1.1 and \( I_0 = \epsilon_0 I_0, p \geq 0 \).

Then there exists a unique global in time solution \( I(\tau, z, y) \) of (19) for \( z = \epsilon x_1 > 0 \). Therefore the functions \( \tilde{\rho}_\epsilon, \tilde{u}_\epsilon \), defined by (14) and (15) in the half space

\[
\{ x_1 > 0, \quad x' \in \mathcal{R}^{n-1}, \quad t > 0 \},
\]

are smooth:

\[
\tilde{\rho}_\epsilon \in C([0, \infty[, H^s(\mathcal{R}/LZ \times \mathcal{R}_{y}^{n-1})) \cap C^1([0, \infty[, H^s-2(\mathcal{R}/LZ \times \mathcal{R}_{y}^{n-1})),
\]

\[
\tilde{u}_\epsilon \in C([0, \infty[, H^{s-2}(\mathcal{R}/LZ \times \mathcal{R}_{y}^{n-1})) \cap C^1([0, \infty[, H^{s-4}(\mathcal{R}/LZ \times \mathcal{R}_{y}^{n-1})).
\]

The Navier–Stokes system (2) in the half space with the initial data (21) and following boundary conditions

\[
(\tilde{u}_\epsilon - u_\epsilon)|_{x_1=0} = 0,
\]

and when the first component of the velocity is positive \( u_\epsilon,1|_{x_1=0} > 0 \) (i.e. at points where the fluid enters the domain) the additional boundary condition

\[
(\tilde{\rho}_\epsilon - \rho_\epsilon)|_{x_1=0} = 0.
\]

When \( u_\epsilon,1|_{x_1=0} \leq 0 \) there is not any boundary condition for \( \rho_\epsilon \).

Suppose also that \( u_\epsilon \to 0, \rho_\epsilon \to \rho_0 \) as \( |x| \to \infty \).

Then there exists a constant \( T_0 > 0 \) such that for all \( t < T_0/\epsilon^{2+p} \) there exists a unique solution \( U_\epsilon = (\rho_\epsilon, u_\epsilon) \) of Navier–Stokes system (2) with the same smoothness as (24), (25).

Remark 4. Since the boundary conditions for the Navier–Stokes system are periodic and of mean value zero on \( t, u_\epsilon,1|_{x_1=0} \) changes the sign and the inflow part of the boundary goes after the inflow one and so on. This avoids the pathology which may result from a change of type of the boundary condition in the tangential variables. This hypothesis follows from the physical works of Zabolotskaya (see [2]), where one takes as the initial conditions for the KZK equation with the help of Definition 2.4.

The pair of functions \( \tilde{\rho}_\epsilon, \tilde{u}_\epsilon \), satisfying the boundary conditions in the half space.

Remark 5. To have the estimate (26) it is sufficient to have an admissible weak solution of the Navier–Stokes system (2) satisfying the boundary conditions in the half space.

Definition 2.4. The pair of functions \( (\rho, u) \) is called an admissible weak solution of Navier–Stokes system (2) satisfying the boundary conditions in the half space if it satisfies the following properties:
(i) it is a weak solution of (2), and it satisfies in the sense of distributions (see [3, p.52])

\[ \partial_t \eta(U_\epsilon) + \nabla \cdot q(U_\epsilon) - \epsilon \nu \begin{bmatrix} 0 \\ u_\epsilon \Delta u_\epsilon \end{bmatrix} \leq 0, \]

(ii) it satisfies the equality

\[
- \int_{x_1>0} \frac{U_\epsilon^2}{2} \, dx + \int \int_{x_1>0} \left( \nabla \cdot U_\epsilon F(U_\epsilon) + \epsilon \nu \left[ \begin{array}{c} 0 \\ |\nabla \cdot u_\epsilon|^2 \end{array} \right] \right) \, dx \, ds + \int_{x_1>0} U_0^2(x) \, dx \\
+ \int \int_{R^{n-1}} U_\epsilon \left( F(U_\epsilon) - \epsilon \nu \left[ \begin{array}{c} 0 \\ \nabla u_\epsilon \end{array} \right] \right) \bigg|_{x_1=0} \, dx' \, dt = 0.
\]

References