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Mathematical Analysis

Gabor frames with Hermite functions

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Abstract

We investigate Gabor frames based on a linear combination of Hermite functions H_n. We derive sufficient conditions on the lattice $\Lambda \subseteq \mathbb{R}^2$ such that the Gabor system $\{e^{2\pi i\lambda_2 t} H_n(t-\lambda_1)\colon \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$ is a frame. An example supports our conjecture that our conditions are sharp. The main tools are growth estimates for the Weierstrass σ-function and a new type of interpolation problem for entire functions on the Bargmann–Fock space. ***To cite this article: K. Gröchenig, Y. Lyubarskii, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Frames de Weyl–Heisenberg et fonctions d'Hermite. Nous étudions les propriétés de frame de l' ensemble $\{e^{2\pi i\lambda_2 t} H_n(t-\lambda_1)\colon \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$, où H_n est une fonction de Hermite et Λ est un réseau dans $\mathbb{R}^2$. Nous donnons des conditions suffisantes sur la densité de Λ pour que la propriété de frame soit satisfaite. Un contre-exemple suggère que nos conditions sont aussi nécessaires. Les outils principaux sont des estimations de croissance pour la fonction σ de Weierstrass et un nouveau type d'interpolation dans l'espace de Bargmann–Fock. ***Pour citer cet article : K. Gröchenig, Y. Lyubarskii, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

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Soit $z = (x, y) \in \mathbb{R}^2$ où $z = x + y\mathrm{i} \in \mathbb{C}$, on pose

$$\pi_z f(t) = e^{2\pi i y t} f(t-x), \quad x, y, t \in \mathbb{R},$$

le décalage en temps-fréquence d'une fonction f. Un réseau $\Lambda \subset \mathbb{R}^2$ est un sous-groupe discret de la forme $\Lambda = A\mathbb{Z}^2$, où A est une matrice inversible. L'aire du rectangle fondamental est $s(\Lambda) = |\det A|$, la densité usuelle de Λ est $s(\Lambda)^{-1}$.

On étudie des conditions sous lesquelles est satisfaite l'équivalence

$$A\|f\|^2_{L^2(\mathbb{R})} \leqslant \sum_{\lambda \in \Lambda} \left|\langle f, \pi_\lambda g\rangle_{L^2(\mathbb{R})}\right|^2 \leqslant B\|f\|^2_{L^2(\mathbb{R})} \quad \forall f \in L^2(\mathbb{R}). \tag{1}$$

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Si (1) est satisfait, on dit que l'ensemble $\mathcal{G}(g,\Lambda)=\{\pi_\lambda g\colon\ \lambda\in\Lambda\}$ est un frame de Gabor (ou frame de Weyl–Heisenberg) pour $L^2(\mathbb{R})$. En général, pour une fonction g arbitraire, la caractérisation de tous les réseaux pour lesquels $\mathcal{G}(g,\Lambda)$ est un frame, est extremement difficile. En fait, une telle caractérisation n'est connue que pour la Gaussienne $g(t)=\mathrm{e}^{-\pi t^2}$ [12,13] et pour le sécant hyperbolique $g(t)=(\cosh at)^{-1}$ [11].

Notre résultat principal traite le cas où g est une fonction de Hermite $H_n(t)=\mathrm{e}^{\pi t^2}(\frac{\mathrm{d}}{\mathrm{d}t})^n(\mathrm{e}^{-2\pi t^2})$:

Théorème 0.1. *Si $s(\Lambda)<\frac{1}{n+1}$, l'ensemble $\mathcal{G}(H_n,\Lambda)$ est un frame pour $L^2(\mathbb{R})$.*

Dans le cas $n=0$, où H_0 est la fonction Gaussienne, on retrouve le résultat classique de [12,13]. En fait, si $n=0$, la condition $s(\Lambda)<1$ est suffisante et nécessaire. Dans le cas $n>0$, on donnera un exemple qui suggère que la croissance de la densité $s(\Lambda)^{-1}$ devrait être nécessaire. En ce sens, le résultat est tout à fait surprenante, parce que l'intuition générale disait que pour chaque fonction « raisonnable » la condition $s(\Lambda)<1$ était suffisante pour que $\mathcal{G}(g,\Lambda)$ soit un frame.

Pour démontrer le théorème, on utilise une caractérisation abstraite des frames de Gabor par les identités de Wexler–Raz, cf. [4]. *Soit $\Lambda^\circ=\frac{1}{s(\Lambda)}\Lambda$ le réseau adjoint de Λ. Alors, $\mathcal{G}(g,\Lambda)$ est un frame de $L^2(\mathbb{R})$, si et seulement si il existe une fonction duale $\gamma\in L^2(\mathbb{R})$ telle que*

$$\langle\gamma,\pi_\mu g\rangle=\delta_{\mu,0}\quad\forall\mu\in\Lambda^\circ,\tag{2}$$

et si $\mathcal{G}(g,\Lambda)$ est une séquence de Bessel (une condition technique qui est presque toujours satisfaite).

L'existence d'une fonction duale est déduite en transposant le problème d'analyse réelle en un problème d'analyse complexe. En utilisant la transformée de Bargmann B, on traduit les conditions (2) en un problème d'interpolation dans l'espace de Bargmann–Fock $\mathcal{F}$ qui est constitué des fonctions entières dotées de la norme

$$\|F\|_{\mathcal{F}}^2:=\iint_{\mathbb{C}}\left|F(\zeta)\right|^2\mathrm{e}^{-\pi|\zeta|^2}\,\mathrm{d}\zeta<\infty.$$

Le lemme suivant se déduit à partir des propriété de la transformée de Bargmann et du fait que $BH_n(z)=c_nz^n$ pour une constante $c_n>0$:

Lemme 0.2. *L'ensemble $\mathcal{G}(H_n,\Lambda)$ est un frame de Gabor pour $L^2(\mathbb{R})$ si et seulement si il existe une fonction $G\in\mathcal{F}$ telle que*

$$\sum_{k=0}^{n}\binom{n}{k}(-\pi\bar{\mu})^kG^{(n-k)}(\mu)=\delta_{\mu,0}\quad\forall\mu\in\Lambda^\circ.\tag{3}$$

Dans ce cas on peut prendre $\gamma=B^{-1}G$ comme fonction duale de H_n.

Remarque 1. Ces identités sont un nouveau type de problème d'interpolation avec des dérivées et ne sont pas standards, parce que les coefficients sont antiholomorphes.

La théorie classique des fonctions elliptiques fournit une fonction qui s'annule sur un réseau [1], la fonction σ de Weierstrass definie par

$$\sigma_{\Lambda^\circ}(u)=u\prod_{\lambda\in\Lambda^\circ\setminus\{0\}}\left(1-\frac{u}{\lambda}\right)\mathrm{e}^{u/\lambda+u^2/(2\lambda^2)}.\tag{4}$$

Alors on pose ($a\in\mathbb{C}$)

$$G(z)=n!^{-1}z^{-1}\sigma_{\Lambda^\circ}(z)^{n+1}\mathrm{e}^{a(n+1)z^2}.$$

Par construction, G satisfait les conditions $G^{(k)}(\mu)=\delta_{\mu,0}\delta_{k,n}$ pour $\mu\in\Lambda^\circ$ et $0\leqslant k\leqslant n$, et par conséquent, (3) est aussi satisfait. Il reste a démontrer que G appartient à l'espace $\mathcal{F}$. Dans le cas $s(\Lambda)=s(\Lambda^\circ)^{-1}<1/(n+1)$ on peut choisir $a\in\mathbb{C}$ tel que la fonction $|G(z)|\mathrm{e}^{-(n+1)|z|^2/s(\Lambda^\circ)}$ soit Λ°-periodique. Alors la conclusion $G\in\mathcal{F}$ découle des estimations très souples de la croissance de σ_{Λ° [9].

1. Introduction

Given a function $g \in L^2(\mathbb{R})$ and a lattice $\Lambda \subset \mathbb{R}^2$, we study the frame property of the set $\{e^{2\pi i \lambda_2 t} g(t-\lambda_1)\colon \lambda = (\lambda_1, \lambda_2) \in \Lambda\}$. We write $\pi_\lambda g = e^{2\pi i \lambda_2 t} g(t-\lambda_1)$ for the time-frequency shift by $\lambda \in \mathbb{R}^2$. Then we call the set $\mathcal{G}(g, \Lambda) = \{\pi_\lambda g\colon \lambda \in \Lambda\}$ a *Gabor frame* or *Weyl–Heisenberg frame*, whenever there exist constants $A, B > 0$ such that, for all $f \in L^2(\mathbb{R})$,

$$A\|f\|^2_{L^2(\mathbb{R})} \leqslant \sum_{\lambda \in \Lambda} \left|\langle f, \pi_\lambda g\rangle_{L^2(\mathbb{R})}\right|^2 \leqslant B\|f\|^2_{L^2(\mathbb{R})}. \tag{5}$$

Gabor frames originate in quantum mechanics through J. von Neumann and in information theory through D. Gabor [7] and nowadays have many applications in signal processing. A large body of results describes the structure of Gabor frames and provides sufficient conditions of a qualitative nature for $\mathcal{G}(g, \Lambda)$ to form a (Gabor) frame, see [3,8] for details and references.

The general problem of characterizing *all* lattices Λ for which $\mathcal{G}(g, \Lambda)$ is a frame seems to be extremely difficult. In fact, this problem is solved only for two classes of basis functions, namely for the Gaussian $H_0(t) = e^{-at^2}$, $a > 0$, in [12,13] and for the hyperbolic secant $g(t) = (\cosh at)^{-1}$ in [11]. For the Gaussian, our understanding is based on the connection between the frame property of $\mathcal{G}(g, \Lambda)$ and a classical interpolation problem in the Bargmann–Fock space of entire functions.

In this Note we study the frame property of $\mathcal{G}(g, \Lambda)$ in the case when g is a Hermite function $g = H_n$ or a finite linear combination of Hermite functions. Although this case seems to be similar to the case of Gaussian Gabor frames, it is not and leads us to a striking result: the critical density of the lattice for $\mathcal{G}(H_n, \Lambda)$ to be a Gabor frame depends on n and increases with n. For $n = 0$ (the case of the Gaussian), we recover (the lattice part of) the results [12,13]. An example for $n = 1$ suggests that the sufficient conditions are sharp for all n. The technical tool is a new type of interpolation problem in the Bargmann–Fock space of entire functions. This problem is not 'purely holomorphic': the values of linear combination of a function and its derivatives are prescribed in the lattice points; however, the coefficients of this combination are antiholomorphic polynomials.

The next section includes the precise setting of the problem, auxiliary facts from time-frequency analysis and complex analysis. In Section 3 we prove the main result and remark that it is sharp for $n = 1$. In a subsequent paper we will give a more detailed analysis and applications to the Balian–Low theorem, Gabor superframes, and the time-frequency analysis of distributions.

2. Tools

2.1. Gabor frames and Bessel sequences

(See [8] for an exposition). A *lattice* is a discrete subgroup in $\mathbb{R}^2$ of the form $\Lambda = A\mathbb{Z}^2$, where A is an invertible 2×2 matrix. The size of Λ is defined as the area of its cell $s(\Lambda) = |\det A|$, the density of Λ is $d(\Lambda) = s(\Lambda)^{-1}$, which equals the Beurling density (see, e.g., [13]). The *adjoint lattice* is $\Lambda^\circ := s(\Lambda)^{-1}\Lambda$ and has size $s(\Lambda^\circ) = s(\Lambda)^{-1}$.

We say that $\mathcal{G}(g, \Lambda) = \{\pi_\lambda g\colon g \in \Lambda\}$ is a *Bessel sequence*, if only the right-hand inequality of (5) is satisfied. The Bessel property of $\mathcal{G}(g, \Lambda)$ is a technical condition and depends both on g and Λ, but it is easy to formulate universal conditions, see Lemma 2.4 below.

The frame property (5) of the system $\mathcal{G}(g, \Lambda)$ is much more delicate. A necessary condition is given by the following density theorem [3,8]:

Theorem 2.1. *If $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R})$, then $s(\Lambda) \leqslant 1$.*

An abstract characterization of Gabor frames is provided by the so-called Wexler–Raz identities [4,10].

Theorem 2.2. *Let $g \in L^2(\mathbb{R})$ and $\Lambda \subset \mathbb{R}^2$ be a lattice with adjoint lattice Λ°. Then the following are equivalent*:

(i) *$\mathcal{G}(g, \Lambda)$ is a Gabor frame for $L^2(\mathbb{R})$.*
(ii) *There exists a function $\gamma \in L^2(\mathbb{R})$ (a dual window), such that $\mathcal{G}(\gamma, \Lambda)$ is a Bessel sequence in L^2 and*

$$\langle \gamma, \pi_\mu g\rangle = \delta_{\mu,0} \quad \forall \mu \in \Lambda^\circ. \tag{6}$$

2.2. Transition to complex analysis

The *Bargmann–Fock space* $\mathcal{F}$ is the Hilbert space $\mathcal{F}$ of entire functions F such that

$$\|F\|_{\mathcal{F}}^2 := \iint_{\mathbb{C}} \left|F(\zeta)\right|^2 \mathrm{e}^{-\pi|\zeta|^2} m_\zeta < \infty, \tag{7}$$

where $\mathrm{d}m_\zeta$ is the Lebesgue measure on $\mathbb{C}$. The corresponding inner product is denoted by $\langle\cdot,\cdot\rangle_{\mathcal{F}}$.

The *Bargmann transform* B of a function f on $\mathbb{R}$ is defined as

$$B: f \mapsto Bf(\zeta) = F(\zeta) = 2^{1/4} \int_{-\infty}^{\infty} f(t)\mathrm{e}^{2\pi t\zeta - \pi t^2 - \pi\zeta^2/2}\,\mathrm{d}t, \quad \zeta \in \mathbb{C}. \tag{8}$$

Proposition 2.3. ([6]) (i) *B is a unitary mapping from $L^2(\mathbb{R})$ onto the Bargmann–Fock space $\mathcal{F}$.*

(ii) *$\mathcal{F}$ is a reproducing kernel Hilbert space. If $e_\zeta(w) = \mathrm{e}^{\pi\bar{\zeta}w}$, then $F(\zeta) = \langle F, e_\zeta\rangle_{\mathcal{F}}$.*

(iii) *Define a 'shift' β_z, $z \in \mathbb{C}$, on $\mathcal{F}$ by $\beta_z F(\zeta) = \mathrm{e}^{\mathrm{i}\pi xy}\mathrm{e}^{-\pi|z|^2/2}\mathrm{e}^{\pi\zeta\bar{z}}F(\zeta - z)$, then B intertwines the shifts π and β in the sense that $\beta_z = B\pi_z B^{-1}$ for all $z \in \mathbb{C}$.*

(iv) *Let $H_n(t) = \mathrm{e}^{\pi t^2}(\frac{\mathrm{d}}{\mathrm{d}t})^n(\mathrm{e}^{-2\pi t^2})$ denote the n-th Hermite function, then $BH_n(z) = \pi^{n/2}n!^{-1/2}z^n$.*

Lemma 2.4. *Assume that g satisfies the condition*

$$\|g\|_{M^1} := \iint_{\mathbb{C}} \left|Bg(\zeta)\right|\mathrm{e}^{-\pi|\zeta|^2/2}\,\mathrm{d}m_\zeta < \infty, \tag{9}$$

then the sequence $\mathcal{G}(g,\Lambda)$ is a Bessel sequence in $L^2(\mathbb{R})$ for each lattice Λ [5].

Remark 1. The M^1-norm in (9) defines the Feichtinger algebra, which occurs in numerous applications in harmonic analysis [5].

2.3. Growth estimates for Weierstrass's σ-functions

We identify $(x,y) \in \mathbb{R}^2$ with $z = x + \mathrm{i}y \in \mathbb{C}$. Then we may write the lattice Λ as $\Lambda = \{m_1\lambda_1 + m_2\lambda_2\colon m_1, m_2 \in \mathbb{Z}\}$ for some basis $\lambda_1, \lambda_2 \in \mathbb{C}$ satisfying $\operatorname{Im}(\lambda_2/\lambda_1) > 0$. The size of Λ is then given by $s(\Lambda) = \frac{1}{2\mathrm{i}}(\bar{\lambda}_1\lambda_2 - \bar{\lambda}_2\lambda_1) = |\lambda_1|^2 \operatorname{Im}(\lambda_2/\lambda_1) > 0$.

The *Weierstrass σ-function* corresponding to the lattice Λ is

$$\sigma_\Lambda(u) = \sigma(u) = u \prod_{\lambda\in\Lambda\setminus\{0\}} \left(1 - \frac{u}{\lambda}\right)\mathrm{e}^{u/\lambda + u^2/(2\lambda^2)}. \tag{10}$$

The theory of elliptic functions furnishes the following properties of σ_Λ, see, e.g., [1].

Proposition 2.5. (i) *σ_Λ is an entire function whose zero set is precisely Λ.*

(ii) *Legendre's formula*: $(\log\sigma(u))''$ *is Λ-periodic, $\eta_k = (\log\sigma(u+\lambda_k) - \log\sigma(u))' = \mathrm{const.}$ and $\eta_1\lambda_2 - \eta_2\lambda_1 = 2\pi\mathrm{i}$.*

(iii) *Multiplicative quasi-periodicity of σ: with $\eta_1, \eta_2 \in \mathbb{C}$ as in* (ii)*, we have*

$$\sigma(u+\lambda_k) = -\sigma(u)\mathrm{e}^{\eta_k u + \frac{1}{2}\eta_k\lambda_k}, \quad k = 1, 2. \tag{11}$$

In order to understand the growth properties of σ_Λ, we follow an argument of Hayman [9] who realized that, after a proper normalization and growth compensation, the absolute value of σ_Λ becomes a doubly periodic function. Let us define the numbers

$$\alpha(\Lambda) = \frac{\mathrm{i}\pi}{\bar{\lambda}_1\lambda_2 - \bar{\lambda}_2\lambda_1} = \frac{\pi}{2s(\Lambda)} \quad \text{and} \quad a(\Lambda) = \frac{1}{2}\frac{\eta_1\lambda_2 - \eta_2\lambda_1}{\bar{\lambda}_1\lambda_2 - \bar{\lambda}_2\lambda_1}. \tag{12}$$

The following result can be deduced from (11), (12) by a straightforward, but lengthy computation.

Theorem 2.6. *The function* $|\sigma_\Lambda(u)\mathrm{e}^{a(\Lambda)u^2}|\mathrm{e}^{-\alpha(\Lambda)|u|^2}$ *is periodic with periods* λ_1 *and* λ_2. *Therefore, the modified sigma-function* $\sigma_\Lambda(u)\mathrm{e}^{a(\Lambda)u^2}$ *satisfies the growth estimate*

$$\left|\sigma_\Lambda(u)\mathrm{e}^{a(\Lambda)u^2}\right| \leqslant C\mathrm{e}^{(\pi/(2s(\Lambda)))|u|^2}. \tag{13}$$

3. Gabor frames with Hermite functions

Our main result provides a sufficient condition for the family $\mathcal{G}(H_n, \Lambda)$ to form a frame for $L^2(\mathbb{R})$. The condition involves only the size of the lattice.

Theorem 3.1. *Let* $n \in \mathbb{Z}, n \geqslant 0$. *If* $s(\Lambda) < (n+1)^{-1}$, *then* $\mathcal{G}(H_n, \Lambda)$ *is a frame for* $L^2(\mathbb{R})$.

In the first step of the proof, we translate the frame property to an interpolation problem in $\mathcal{F}$.

Given a (real-valued) window function g we consider the short time Fourier transform (STFT) V_g as a function of a complex variable $z = x + \mathrm{i}y$ instead of 'time' $x \in \mathbb{R}$ and 'frequency' $y \in \mathbb{R}$:

$$V_g f(z) = \langle f, \pi_z g\rangle_{L^2} = \int_{-\infty}^{\infty} f(t)g(t-x)\mathrm{e}^{-2\mathrm{i}\pi yt}\,\mathrm{d}t.$$

The next proposition details a formula for the STFT with respect to a Hermite function:

Proposition 3.2. *Let* $f \in L^2(\mathbb{R})$, $F = Bf$, *and* H_n *be the n-th Hermite function. Then*

$$V_{H_n} f(z) = \mathrm{e}^{\mathrm{i}\pi xy}\mathrm{e}^{-\pi|z|^2/2}\frac{1}{\sqrt{\pi^n n!}}\sum_{k=0}^{n}\binom{n}{k}(-\pi\bar{z})^k F^{(n-k)}(z). \tag{14}$$

Proof. With the properties of the Bargmann transform of Proposition 2.3, we obtain

$$\begin{aligned} V_{H_n} f(z) &= \langle f, \pi_z H_n\rangle_{L^2} = \langle F, \beta_z B H_n\rangle_{\mathcal{F}} = \kappa_n \mathrm{e}^{\mathrm{i}\pi xy}\mathrm{e}^{-\pi|z|^2/2}\left\langle F(w), \mathrm{e}^{\pi\bar{z}w}(w-z)^n\right\rangle_{\mathcal{F}} \\ &= \kappa_n \mathrm{e}^{\mathrm{i}\pi xy}\mathrm{e}^{-\pi|z|^2/2}\sum_{k=0}^{n}\binom{n}{k}(-\bar{z})^k\left\langle F(w), w^{n-k}\mathrm{e}^{\pi\bar{z}w}\right\rangle_{\mathcal{F}}. \end{aligned}$$

Proposition 2.4(ii) yields $\langle F(w), w^{n-k}\mathrm{e}^{\pi\bar{z}w}\rangle_{\mathcal{F}} = \pi^{k-n}F^{(n-k)}(z)$ and thus (14) follows. □

Combining this proposition with Theorem 2.2, we obtain the following criterion for Gabor frames with Hermite functions. We are lead to a new type of interpolation problem in the Bargmann–Fock space:

Proposition 3.3. *Let* $\Lambda \subset \mathbb{R}^2$ *be a lattice and* $n \geqslant 0$. *The set* $\mathcal{G}(H_n, \Lambda)$ *is a* (*Gabor*) *frame for* $L^2(\mathbb{R})$ *if and only if there exists a function* $\gamma \in L^2(\mathbb{R})$ *such that* $\mathcal{G}(\gamma, \Lambda)$ *is a Bessel sequence and the Bargmann transform* $G = B\gamma$ *satisfies*

$$\sum_{k=0}^{n}\binom{n}{k}(-\pi\bar{\mu})^k G^{(n-k)}(\mu) = \delta_{\mu,0} \quad \forall\mu \in \Lambda^\circ. \tag{15}$$

Remark. Amazingly enough, almost the same interpolation problem appears in the study of the Landau equation and its eigenspaces corresponding to high energy levels. The zero energy level is investigated in [2] and leads to a standard uniqueness and interpolation problem in the Bargmann–Fock space.

3.1. Sketch of the proof of Theorem 3.1

Let σ_{Λ° be the σ-function corresponding to the adjoint lattice Λ° of Λ. Denote

$$G(z) = n!^{-1} z^{-1} \sigma_{\Lambda^\circ}(z)^{n+1} \mathrm{e}^{a(\Lambda^\circ)(n+1)z^2}. \tag{16}$$

It follows from (13) that

$$\left|G(z)\right| \leqslant \text{Const.}\, \mathrm{e}^{\pi k|z|^2/2} \quad \text{with } k = \frac{n+1}{s(\Lambda^\circ)} = (n+1)s(\Lambda) < 1. \tag{17}$$

This growth estimates implies immediately that $G \in \mathcal{F}$. By construction, the derivatives of G satisfy $G^{(k)}(\mu) = \delta_{\mu,0}\delta_{k,n}$ for $\mu \in \Lambda^\circ$ and $0 \leqslant k \leqslant n$, and thus interpolation problem (15) is solved.

Now we choose the dual window to be $\gamma = B^{-1}G$. The growth estimate (17) yields $\gamma \in M^1$, and Lemma 2.4 implies that $\mathcal{G}(\gamma, \Lambda)$ is a Bessel sequence. By Proposition 3.3 $\mathcal{G}(H_n, \Lambda)$ is a frame. □

Remarks. 1. It can be shown that $|\gamma(t)| \leqslant \text{Const.}\, \mathrm{e}^{-\epsilon t^2}$ and $|\hat{\gamma}(\omega)| \leqslant \text{Const.}\, \mathrm{e}^{-\epsilon\omega^2}$ for each $\epsilon < (1-k)/2$.

2. For $n = 0$ we recover the sharp result that $\mathcal{G}(H_0, \Lambda)$ is a frame if and only if $s(\Lambda) < 1$ from [12,13].

A similar argument works for Gabor systems generated by a linear combination of Hermite functions:

Proposition 3.4. *Let $n \in \mathbb{Z}, n \geqslant 0$ and $g = \sum_{k=0}^{n} b_k H_k$, $b_n \neq 0$. If $s(\Lambda) < (n+1)^{-1}$, then $\mathcal{G}(g, \Lambda)$ is a frame for $L^2(\mathbb{R})$. The dual window γ may be chosen as in* (16) *for H_n.*

3.2. A counter-example

The function G that we constructed in the proof of Theorem 3.1 meets much stronger restrictions than just (15), namely $G^{(k)}(\mu) = \delta_{\mu,0}\delta_{k,n}$ for $\mu \in \Lambda^\circ$ and $0 \leqslant k \leqslant n$. The following statement provides some evidence that Theorem 3.1 is optimal, but it is only a partial converse:

Proposition 3.5. *If $\Lambda = \alpha\mathbb{Z} \times \beta\mathbb{Z}$ and $\alpha\beta = 1/2$, then $\mathcal{G}(H_{2n+1}, \Lambda)$ is not a Gabor frame.*

Proof. We use a Zak transform argument. For $\alpha > 0$ the *Zak transform* is defined by $Z_\alpha f(x,\xi) = \sum_{k\in\mathbb{Z}} f(x - \alpha k)\mathrm{e}^{2\mathrm{i}\pi\alpha k\xi}$. If $Z_\alpha g$ is continuous and $\mathcal{G}(g, \alpha\mathbb{Z} \times \frac{1}{2\alpha}\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$, then the functions $Z_\alpha g(x,\xi)$ and $Z_\alpha g(x - \alpha/2, \xi)$ cannot have a common zero, see, e.g., [8, p. 159].

However, since $Z_\alpha H_{2n+1}$ is a continuous function and since H_{2n+1} is odd, we find that $Z_\alpha H_{2n+1}(0,0) = 0$ and $Z_\alpha H_{2n+1}(\alpha/2, 0) = 0$, violating the necessary condition just described. □

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