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C. R. Acad. Sci. Paris, Ser. I 344 (2007) 215-218

COMPTES RENDUS MATHEMATIQUE

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Group Theory

On the set of covolumes of lattices for Fuchsian buildings

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Received 25 July 2006; accepted after revision 6 December 2006

Available online 17 January 2007

Presented by Étienne Ghys

Abstract

We construct a nonuniform lattice and an infinite family of uniform lattices in the automorphism group of a Fuchsian building. We use complexes of groups and basic facts about spherical buildings. A consequence is that the set of covolumes of lattices for this building is nondiscrete. *To cite this article: A. Thomas, C. R. Acad. Sci. Paris, Ser. I 344 (2007).* © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

Résumé

Sur l'ensemble des covolumes des réseaux pour les immeubles fuchsiens. Nous construisons un réseau noncocompact et une famille infinie de réseaux cocompacts dans le groupe d'automorphismes d'un immeuble fuchsien. Nous utilisons des complexes de groupes et quelques propriétés élémentaires des immeubles sphériques. Une conséquence est que l'ensemble des covolumes des réseaux pour cet immeuble est nondiscret. *Pour citer cet article : A. Thomas, C. R. Acad. Sci. Paris, Ser. I 344 (2007).* © 2007 Académie des sciences. Published by Elsevier Masson SAS. All rights reserved.

1. Introduction

Let Δ be the finite building of rank 2 associated to a Chevalley group. A (k, Δ) -building is a hyperbolic polygonal complex X, with the link at each vertex Δ , and each 2-cell a regular hyperbolic k-gon. Let Aut(X) be the group of cellular isometries of X. Since X is locally finite, Aut(X), with the compact-open topology, is locally compact. Let μ be a Haar measure on G = Aut(X). A discrete subgroup $\Gamma \leq G$ is a *lattice* if $\mu(\Gamma \setminus G)$ is finite, and Γ is *uniform* if $\Gamma \setminus G$ is compact.

Very few lattices in Aut(X) are known. Some (k, Δ) -buildings are Kac–Moody buildings, so have a nonuniform lattice (Carbone and Garland, [3]; Rémy, [6]). Haglund [5] constructed Coxeter lattices for k even. Bourdon [1] and Gaboriau and Paulin [4] constructed uniform lattices using polygons of groups, and Bourdon constructed uniform and nonuniform lattices for certain (k, Δ) -buildings, k = 4 or 5, by lifting lattices for affine buildings (Example 1.5.2 of [1]).

Using complexes of groups, for each $k \ge 4$ divisible by 4, and each Δ , we construct a nonuniform lattice and an infinite family of uniform lattices for the unique locally reflexive (k, Δ) -building X with trivial holonomy (see

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below for definitions). The construction applies the Levi decomposition and basic facts about spherical buildings. A consequence is that the set of covolumes of lattices for X is nondiscrete.

2. Preliminaries

Let X be a (k, Δ) -building and G = Aut(X). The following characterisation of lattices in G is the same as Proposition 1.4.2 of [1], except that we consider the action on vertices rather than on 2-cells:

Proposition 2.1. Suppose $G \setminus X$ is compact. Let $\Gamma \leq G$ act properly discontinuously on X and let V be a set of representatives of the vertices of $\Gamma \setminus X$. Then Γ is a lattice if and only if the series

$$\sum_{v \in \mathcal{V}} \frac{1}{|\Gamma_v|} \tag{1}$$

converges, and Γ is uniform if and only if $\Gamma \setminus X$ is compact.

The Haar measure μ on G may be normalised so that $\mu(\Gamma \setminus G)$ equals the series (1) (Serre, [8]).

We next state local conditions for the universal cover of a complex of groups to be a (k, Δ) -building (see [2] for Haefliger's theory of complexes of groups). Let *Y* be a (multi)-simplicial complex of dimension 2, with a colouring of vertices by {0, 1, 2} which is injective on each 2-simplex. We say a vertex of *Y* is an *n*-vertex, for n = 0, 1, 2, if it has colour *n*. The following is an easy generalisation of Theorem 0.1 of [4]:

Theorem 2.2. Suppose G(Y) is a complex of groups over Y, such that the local development at each n-vertex of Y is: for n = 0, the barycentric subdivision of Δ ; for n = 1, the complete bipartite graph $K_{2,s}$, with s the valence of a vertex of Δ ; and for n = 2, the barycentric subdivision of a regular hyperbolic k-gon. Then G(Y) is developable, with universal cover (the barycentric subdivision of) a (k, Δ) -building.

For fixed (k, Δ) , there may be uncountably many (k, Δ) -buildings (see, for example, Theorem 0.2 of [4]). We now recall conditions, due to Haglund in [5], under which local data do specify the building. For each (closed) edge *a* of *X*, let $\mathcal{U}(a)$ be the union of the (closed) 2-cells of *X* which meet *a*. Then *X* is *locally reflexive* if every $\mathcal{U}(a)$ has a *reflection*, that is, an automorphism of order 2 which exchanges the ends of *a*, and preserves each 2-cell containing *a*. For each 2-cell *C* of *X*, let the edges of *C* be cyclically labelled by a_1, \ldots, a_k , and let v_i be the vertex of *C* contained in the edges a_{i-1} and a_i . A locally reflexive building *X* has *trivial holonomy* if for each 2-cell *C*, there is a set of reflections $\sigma_1, \ldots, \sigma_k$ of $\mathcal{U}(a_1), \ldots, \mathcal{U}(a_k)$, such that the composition $\sigma_k \circ \cdots \circ \sigma_1$ (where each σ_i is restricted to the link of the vertex v_i) is the identity. Finally, *X* is *homogeneous* if Aut(*X*) acts transitively on the set of vertices of *X*.

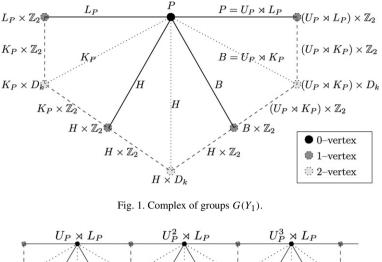
Theorem 2.3. (Haglund, [5]) Let $k \ge 4$ be even. Then there exists a unique locally reflexive (k, Δ) -building X with trivial holonomy, and X is homogeneous.

3. Construction of lattices

Let Δ be the spherical building of rank 2 associated to a finite Chevalley group \mathcal{G} . Then Δ is a generalised *m*-gon, that is, a bipartite graph with diameter *m* and girth 2m, for $m \in \{3, 4, 6, 8\}$ (see [7]). Let *B* be the Borel subgroup of \mathcal{G} and let *P* be a standard parabolic subgroup of \mathcal{G} . Recall that \mathcal{G} acts on Δ by type-preserving automorphisms, hence so does *P*.

Lemma 3.1. The quotient graph $P \setminus \Delta$ is a ray of m edges. Moreover, there are subgroups U_P , L_P and $K_P < H_1 < \cdots < H_{m-2} < B$ of P such that the quotient graph of groups for the action of P on Δ is:

Proof. The group *P* is the stabiliser in \mathcal{G} of a vertex *v* of Δ , and *B* is the stabiliser of an edge containing *v*. Since \mathcal{G} acts transitively on pairs of vertices at fixed distance in Δ , *P* acts transitively on the sets of vertices of Δ at distances



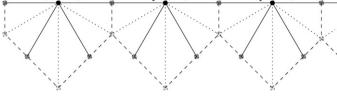


Fig. 2. Sketch of $G(Y_{\infty})$.

 $j = 1, 2, ..., m = \text{girth}(\Delta)$ from v. Hence the quotient $P \setminus \Delta$ is a ray of m edges, with L_P , K_P and the H_i the subgroups of P stabilising vertices and edges of Δ as shown.

By Theorem 6.18 of [7], there is a subgroup U_P of P such that $P = U_P \rtimes L_P$. We now show $B = U_P \rtimes K_P$. By definition of U_P (see [7]), we have $U_P < B < P$, thus $U_P < B$. As $K_P < L_P$ and $U_P \cap L_P = 1$, it follows that $U_P \cap K_P = 1$. Vertices at distance m in Δ have the same valence (Exercise 6.3 of [7]), so

$$[L_P:K_P] = [P:B] = [U_P L_P:B].$$

Hence $|B| = |U_P||K_P|$ and so $B = U_P K_P$. We conclude that $B = U_P \rtimes K_P$. \Box

Assume $k \ge 4$ is even and consider the complex of groups $G(Y_1)$ shown in Fig. 1, for the case m = 3, with $H = H_1 = H_{m-2}$. To construct $G(Y_1)$, we first take the barycentric subdivision of the ray of groups above, with naturally defined local groups. We then form the simplicial cone on this ray, assign P to the cone point, and extend to the boundary using direct products with copies of \mathbb{Z}_2 the cyclic group of order 2, and D_k the dihedral group of order k, with each D_k generated by the two adjacent copies of \mathbb{Z}_2 . The construction of $G(Y_1)$ for other values of m is similar: there are m 2-vertices with groups $K_P \times D_k$, $H_i \times D_k$ for $1 \le i \le m - 2$, and $(U_P \rtimes K_P) \times D_k$.

For $n \ge 1$ let U_P^n be the direct product of n copies of U_P . Since U_P is normal in P, for any subgroup Q of P we may form $U_P^n \rtimes Q$, with the action on each copy of U_P by conjugation in P. Hence, in Fig. 1, we may replace each copy of $B = U_P \rtimes K_P$ and $P = U_P \rtimes L_P$ by respectively $U_P^n \rtimes K_P$ and $U_P^n \rtimes L_P$, and each L_P , K_P and H by respectively $U_P^{n-1} \rtimes L_P$, $U_P^{n-1} \rtimes K_P$ and $U_P^{n-1} \rtimes H$ (and similarly for other values of m). Call the resulting complex of groups $G(Y_n)$. The local developments are unchanged since U_P^{n-1} is a common normal subgroup.

Assume k is divisible by 4. As sketched in Fig. 2 (for the case m = 3, and showing only the 0-vertex groups) we may form a complex of groups $G(Y_{\infty})$ by "gluing" together $G(Y_1)$, $G(Y_2)$, and so on. More precisely, for $n \ge 1$, we identify the cells of $G(Y_n)$ and $G(Y_{n+1})$ with local groups $(U_P^n \rtimes L_P) \times \mathbb{Z}_2$, $(U_P^n \rtimes K_P) \times \mathbb{Z}_2$ and $(U_P^n \rtimes K_P) \times D_k$. We then remove the \mathbb{Z}_2 -factors and replace D_k by $D_{k/2}$ (since k is divisible by 4, $\frac{k}{2}$ is even; if k = 4 then $D_2 \cong \mathbb{Z}_2$ and the images of the two adjacent copies of \mathbb{Z}_2 coincide).

By Lemma 3.1 and Theorem 2.2, the universal cover X of $G(Y_{\infty})$ is a (k, Δ) -building. The direct products with \mathbb{Z}_2 at 1-vertices induce canonical reflections along certain edges of X. The remaining edges of X cover edges of Y_{∞} which join 0-vertices, and it is not hard to construct reflections here too. Hence X is locally reflexive. The local

developments at 2-vertices are obtained using dihedral groups whose reflections are the local reflections. Thus the holonomy is trivial.

Let $\Gamma = \pi_1(G(Y_\infty))$. Then $N = \ker(\Gamma \to G = \operatorname{Aut}(X))$ is contained in each local group of $G(Y_\infty)$, so has bounded order. Abusing notation, we identify Γ and Γ/N . By Theorem 2.3, X is homogeneous, so $G \setminus X$ is compact. Since $G(Y_\infty)$ has finite local groups, Γ acts properly discontinuously. Thus, as the series

$$\sum_{v \in \mathcal{V}} \frac{1}{|\Gamma_v|} = \sum_{n=1}^{\infty} \frac{1}{|U_P^n \rtimes L_P|} = \frac{1}{|L_P|} \sum_{n=1}^{\infty} \frac{1}{|U_P|^n}$$
(2)

is convergent, Proposition 2.1 implies that Γ is a nonuniform lattice in G.

An infinite family of uniform lattices, with covolumes the partial sums of (2), is obtained by, for each $n \ge 1$, gluing together $G(Y_1), \ldots, G(Y_n)$. Hence the set of covolumes of lattices in *G* is nondiscrete.

Acknowledgements

I thank J. Alperin and T. Januszkiewicz for very helpful discussions, B. Farb and G.C. Hruska for encouragement and comments on this paper, M. Ronan for answering a question on spherical buildings, and the anonymous referee for many worthwhile suggestions.

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