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Chebyshevian splines: interpolation and blossoms

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Abstract

We state and discuss a theorem which links the existence of blossoms in a spline space (with sections in different Extended Chebyshev spaces and with connection matrices which are not necessarily totally positive) with the possibility of Hermite interpolation in its derivative space under Schoenberg–Whitney conditions. ***To cite this article: A. Kayumov, M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Splines de Chebyshev : interpolation et floraisons. Cette note établit un lien fondamental entre existence de floraisons dans un espace de splines (à sections dans différents espaces de Chebyshev généralisés et avec matrices de connexion non nécessairement totalement positives) et possibilité d'interpoler au sens d'Hermite sous conditions de Schoenberg–Whitney. ***Pour citer cet article : A. Kayumov, M.-L. Mazure, C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Version française abrégée

Floraisons et interpolation d'Hermite sont intimement liées comme le rappelle le théorème suivant :

Théorème 1.1. *Étant donné un W-espace $\mathbb{E}$ de fonctions suffisamment différentiables supposé contenir les constantes, les propriétés suivantes sont équivalentes* :

(1) *les floraisons existent dans l'espace* $\mathbb{E}$;
(2) *dans l'espace dérivé* $D\mathbb{E}$*, tout problème d'interpolation d'Hermite a une unique solution.*

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De plus, l'une ou l'autre de ces propriétés implique la suivante :

(3) *dans l'espace* $\mathbb{E}$ *lui-même, tout problème d'interpolation d'Hermite a une unique solution.*

Le présent travail a été motivé par le désir naturel d'établir le lien analogue dans le cadre d'espaces de splines très généraux. Notre résultat principal est le suivant :

Théorème 1.2. *Étant donné un espace* $\mathbb{S}$ *de W-splines supposé contenir les constantes, les propriétés suivantes sont équivalentes* :

(1) *les floraisons existent dans* $\mathbb{S}$;
(2) *dans l'espace dérivé* $D\mathbb{S}$ *ainsi que dans tout espace de splines obtenu à partir de* $D\mathbb{S}$ *par insertion de noeuds et/ou restriction, un problème d'interpolation d'Hermite donné a une unique solution si et seulement si il satisfait les conditions de Schoenberg–Whitney.*

De plus, l'une ou l'autre de ces propriétés implique la suivante :

(3) *dans l'espace* $\mathbb{S}$ *lui-même ainsi que dans tout espace de splines obtenu à partir de* $\mathbb{S}$ *par insertion de noeuds et/ou restriction, un problème d'interpolation d'Hermite donné a une unique solution si et seulement si il satisfait les conditions de Schoenberg–Whitney.*

Les différentes notions seront définies avec précision dans le corps de la note. Signalons seulement que les propriétés (2) et (3) du Théorème 1.1 peuvent être formulées de façon équivalente sous la forme : $D\mathbb{E}$ (respectivement $\mathbb{E}$) est un espace de Chebyshev généralisé sur l'intervalle considéré. Il est intéressant de noter que, par l'intermédiaire des floraisons, le Théorème 1.2 fournit pour l'interpolation d'Hermite une caractérisation géométrique intrinsèque aux espaces de splines en question. Nous attirons l'attention du lecteur sur le fait que ces derniers sont à sections dans des espaces différents et comportent éventuellement des matrices de connexion. Par ailleurs, notre résultat, totalement exempt d'une quelconque hypothèse de totale positivité, permet d'aller bien au-delà du cadre voisin mais non intrinsèque envisagé dans [7].

1. Introduction

The following theorem linking the existence of blossoms in a W-space with the possibility of Hermite interpolation in its derivative space can by now be regarded as classical. (All the relevant definitions will be recalled in the following section. We will everywhere be working on a bounded real interval $I = [\alpha, \beta]$.)

Theorem 1.1. ([6], Corollary 4.1) *Let* $\mathbb{E}$ *be a W-space on* I *containing constants. Then the following two properties are equivalent*:

(1) *blossoms exist in* $\mathbb{E}$;
(2) *any Hermite interpolation problem has a unique solution in* $D\mathbb{E}$.

From any of the above properties follows:

(3) *any Hermite interpolation problem has a unique solution in* $\mathbb{E}$.

In our work we have been motivated by the desire to derive an analogue of the above theorem for spline spaces. Our main result is the following:

Theorem 1.2. *Let* $\mathbb{S}$ *be a W-spline space containing constants* (*see Definition* 2.2). *Then the following two properties are equivalent*:

(1) *blossoms exist in* $\mathbb{S}$;
(2) *both in* $D\mathbb{S}$ *and in any spline space obtained from it by a combination of knot insertion and restriction to a subinterval an Hermite interpolation problem has a unique solution if and only if it satisfies the Schoenberg–Whitney conditions.*

From any of the above properties follows:

(3) *both in* $\mathbb{S}$ *and in any spline space obtained from it by a combination of knot insertion and restriction to a subinterval an Hermite interpolation problem has a unique solution if and only if it satisfies the Schoenberg–Whitney conditions.*

Note that any of the above conditions (1) and (2) implies that the W-spline space in question is actually an EC-spline space (i.e., a space of splines with sections in EC-spaces, see definition below).

2. Definitions

We now proceed to give the pertinent definitions.

2.1. W-spaces, EC-spaces, and blossoms

Let $\mathbb{E} \subset C^n(I)$ be an $(n+1)$-dimensional space. We say that it is *a W-space on* I if any Taylor interpolation problem in $n+1$ data has a unique solution in $\mathbb{E}$; or, equivalently, if the Wronskian of a basis of $\mathbb{E}$ does not vanish on I. We say that $\mathbb{E}$ is *an Extended Chebyshev space* (*EC-space*) *on* I if any Hermite interpolation problem in $n+1$ data has a unique solution in $\mathbb{E}$; or, equivalently, if the determinant of the collocation matrix of a basis of $\mathbb{E}$ does not vanish whatever the choice of $n+1$ (possibly repeated) collocation points in I.

Note that the second and third conditions of Theorem 1.1 can be thus reformulated by simply saying that $D\mathbb{E}$ and $\mathbb{E}$ are respectively n- and $(n+1)$-dimensional EC-spaces on I.

Let us define blossoms in a W-space $\mathbb{E}$, thus clarifying the meaning of the first condition in Theorem 1.1. Choose n functions $\Phi_1, \ldots, \Phi_n \in \mathbb{E}$ such that $(\mathbb{1}, \Phi_1, \ldots, \Phi_n)$ forms a basis of $\mathbb{E}$; define the function $\Phi = (\Phi_1, \ldots, \Phi_n)$. This is a *mother-function* of the space $\mathbb{E}$, from which any function $F \in \mathbb{E}^d$ (for any $d \geqslant 1$) can be obtained by an affine map. The *osculating flat* $\mathrm{Osc}_i \Phi(x)$ of order i $(0 \leqslant i \leqslant n)$ at a point $x \in I$ is the affine flat defined by the point $\Phi(x)$ and the span of $\{\Phi'(x), \ldots, \Phi^{(i)}(x)\}$.

We say that *blossoms exist in the space* $\mathbb{E}$ if, for any $\tau_1 < \cdots < \tau_r$ in I and any positive integers $\nu_1, \ldots, \nu_r$ such that $\sum_{i=1}^r \nu_i = n$, the osculating flats $(\mathrm{Osc}_{n-\nu_i} \Phi(\tau_i))_{i=1}^r$ intersect at a single point. The *blossom* φ of Φ is then the symmetric function

$$\varphi = (\varphi_1, \ldots, \varphi_n) : I^n \longrightarrow \mathbb{R}^n, \quad \{\varphi(\tau_1{}^{[\nu_1]}, \ldots, \tau_r{}^{[\nu_r]})\} = \bigcap_{i=1}^r \mathrm{Osc}_{n-\nu_i} \Phi(\tau_i),$$

where $t^{[k]}$ stands for a k-fold repetition of the point t.

2.2. W-spline spaces and blossoms

Let us denote by

- $\alpha < t_1 < \cdots < t_q < \beta$ — interior knots;
- $(m_k)_{k=1}^q$, $0 \leqslant m_k \leqslant n$, — knot multiplicities, $m = \sum_{k=1}^q m_k$;
- $\mathcal{K} = (t_1{}^{[m_1]}, \ldots, t_q{}^{[m_q]}) = (\xi_1, \ldots, \xi_m)$ — knot vector;
- $(M_k)_{k=1}^q$ — connection matrices. For each k, M_k is a square matrix of order $n - m_k$ which is lower-triangular and has positive diagonal elements.
- $(\mathbb{E}_k)_{k=0}^q$ — the section spaces. For each k, $\mathbb{E}_k$ is an $(n+1)$-dimensional W-space on $[t_k, t_{k+1}]$ ($t_0 := \alpha$, $t_{q+1} := \beta$) containing constants.

With this, we define the *spline space* $\mathbb{S}$ as the space of all continuous functions $S : I \to \mathbb{R}$ such that

(1) for $k = 0, \dots, q$: the restriction of S to $[t_k, t_{k+1}]$ belongs to $\mathbb{E}_k$;
(2) for $k = 1, \dots, q$: the following connection condition is fulfilled:

$$\big(S'\big(t_k^+\big), \dots, S^{(n-m_k)}\big(t_k^+\big)\big)^T = M_k \cdot \big(S'\big(t_k^-\big), \dots, S^{(n-m_k)}\big(t_k^-\big)\big)^{\mathrm{T}}.$$

We now define blossoms in the case of a W-spline space $\mathbb{S}$, thus clarifying the meaning of the first condition in our theorem. The main difference between the non-spline case of a single space and the spline case is in the domain of definition of blossoms: while in the former case blossoms are defined on the whole of I^n, in the latter case their natural domain of definition is the set $\mathcal{A}$ of *admissible n-tuples*:

$$\mathcal{A} = \big\{(x_1, \dots, x_n) \in I^n \mid \forall i = 1, \dots, q : \min(x_1, \dots, x_n) < t_i < \max(x_1, \dots, x_n) \Rightarrow \\ t_i \text{ appears at least } m_i \text{ times among } (x_1, \dots, x_n)\big\}.$$

The rest of the definition proceeds along the same lines. Due to its definition, the spline space $\mathbb{S}$ contains constants. We can thus choose a basis $(\mathbb{1}, \Sigma_1, \dots, \Sigma_{n+m})$ of $\mathbb{S}$ and consider the mother-function $\Sigma = (\Sigma_1, \dots, \Sigma_{n+m})$. We say that *blossoms exist in the spline space* $\mathbb{S}$ if, for any $\tau_1 < \cdots < \tau_r$ in I and any positive integers $\nu_1, \dots, \nu_r$ such that $\sum_{i=1}^r \nu_i = n$ and such that the n-tuple $({\tau_1}^{[\nu_1]}, \dots, {\tau_r}^{[\nu_r]})$ is admissible, the osculating flats $(\mathrm{Osc}_{n-\nu_i} \Sigma(\tau_i))_{i=1}^r$ intersect at a single point. The *blossom* σ of Σ is then the symmetric function

$$\sigma = (\sigma_1, \dots, \sigma_{n+m}) : \mathcal{A} \longrightarrow \mathbb{R}^{n+m}, \qquad \big\{\sigma\big({\tau_1}^{[\nu_1]}, \dots, {\tau_r}^{[\nu_r]}\big)\big\} = \bigcap_{i=1}^r \mathrm{Osc}_{n-\nu_i} \Sigma(\tau_i).$$

It is due to the admissibility and to the nature of the connection matrices that all osculating flats involved in this definition are well-defined, except possibly for the first and last ones, which, if necessary, must be interpreted as $\mathrm{Osc}_{n-\nu_1} \Sigma(\tau_1^+)$ and $\mathrm{Osc}_{n-\nu_r} \Sigma(\tau_r^-)$ respectively.

We define the *derivative spline space* to be $D\mathbb{S} = \{DS: S \in \mathbb{S}\}$, where D stands for ordinary differentiation, left or right at the knots. Instead of $[\alpha, \beta]$, the domain of definition of the spline functions in $D\mathbb{S}$ is somewhat untraditionally the union $\bigcup_{k=0}^q [t_k^+, t_{k+1}^-]$. We write $DS(t_k^-)$ and $DS(t_k^+)$ to distinguish between the values of a function DS at t_k as a member of $[t_{k-1}, t_k]$ and $[t_k, t_{k+1}]$ respectively.

It is easily seen that $\dim \mathbb{S} = n + 1 + m$ and $\dim D\mathbb{S} = n + m$.

2.3. The Hermite interpolation

We now specify what exactly is meant by an Hermite interpolation problem in the spline case, as well as by the Schoenberg–Whitney conditions, thus clarifying the meaning of the second and third conditions in our theorem.

Given

- $\alpha \leqslant x_1 < \cdots < x_r \leqslant \beta$ — interpolation sites;
- positive integers $\mu_1, \dots, \mu_r \leqslant n$ — interpolation multiplicities, such that $\sum_{k=1}^r \mu_k = \dim \mathbb{S}$ $(= n + m + 1)$ and such that if $x_i = t_k$ for some interior knot t_k, then $\mu_i - 1 \leqslant n - m_k$;
- $\mathcal{I} = ({x_1}^{[\mu_1]}, \dots, {x_r}^{[\mu_r]}) = (y_{-n}, \dots, y_m)$ — the interpolation point vector;
- real numbers $a_{i,j}$, $i = 1, \dots, r$, $j = 0, \dots, \mu_i - 1$ — interpolation data,

the associated *Hermite interpolation problem in* $\mathbb{S}$ is the problem of finding an element $S \in \mathbb{S}$ such that

$$\text{for all } i = 1, \dots, r;\ j = 0, \dots, \mu_i - 1: \quad S^{(j)}\big({x_i}^{\varepsilon_i}\big) = a_{i,j}, \quad \text{where } \varepsilon_i \in \{+, -\} \text{ for } i = 1, \dots, r.$$

Of course, ε_i can be suppressed either when x_i is not a knot or when $j = 0$. An Hermite interpolation problem is thus determined by the interpolation point vector, interpolation data, and the sequence $(\varepsilon_i)_{i=1}^r$. An Hermite interpolation problem in $D\mathbb{S}$ is defined similarly, with $\dim D\mathbb{S}$ substituted in place of $\dim \mathbb{S}$ (in this case, when x_i is a knot, ε_i can no longer be suppressed for $j = 0$, since in contrast to $\mathbb{S}$, splines in $D\mathbb{S}$ do not have to be continuous).

We say that an Hermite interpolation problem in $\mathbb{S}$ *satisfies the Schoenberg–Whitney conditions*, if the knot vector $\mathcal{K}$ and the interpolation point vector $\mathcal{I}$ satisfy the following interlacing condition: for all $k = 1, \dots, m$: $y_{k-n-1} < \xi_k < y_k$.

3. Sketch of the proof

A vital role is played in the proof of our theorem by the existence of bases of minimally supported splines both in $\mathbb{S}$ and in $D\mathbb{S}$. We therefore proceed to give the following:

Definition 3.1 *(B-spline basis).* Given splines $N_k \in \mathbb{S}$, $-n \leqslant k \leqslant m$, we introduce the following properties, which make use of the extended knot vector $(t_0{}^{[n+1]}, t_1{}^{[m_1]}, \dots, t_q^{[m_q]}, t_{q+1}^{[n+1]}) = (\xi_{-n}, \dots, \xi_{m+n+1})$:

$(\mathrm{BSB})_1$ (*support property*) for each $k = -n, \dots, m$: $\operatorname{supp} N_k = [\xi_k, \xi_{k+n+1}]$;
$(\mathrm{BSB})_2$ (*end-point property*) for each $k = -n, \dots, m$: N_k vanishes exactly $n - m_l + l_{\mathcal{K}}(k) + 1$ times at the left end-point $\xi_k = t_l$ of its support and exactly $n - m_r + r_{\mathcal{K}}(k+n+1) + 1$ times at the right end-point $\xi_{k+n+1} = t_r$ of its support, where the counting functions are defined by

$$l_{\mathcal{K}}(i) = \max\{j \colon \xi_{i-j} = \xi_i\}, \qquad r_{\mathcal{K}}(i) = \max\{j \colon \xi_{i+j} = \xi_i\};$$

$(\mathrm{BSB})_3$ (*positivity property*) for each $k = -n, \dots, m$: N_k is positive in the interior of its support;
$(\mathrm{BSB})_4$ (*normalisation property*) $\sum_{k=-n}^{m} N_k(x) = 1$ for all $x \in I$.

Note that in case all knots are simple (i.e. all $m_k = 1$) property $(\mathrm{BSB})_2$ is redundant. If properties $(\mathrm{BSB})_1$–$(\mathrm{BSB})_4$ are satisfied, we say that the sequence $(N_k)_{k=-n}^{m}$ is *the B-spline basis of* $\mathbb{S}$. This terminology is justified by the fact that as soon as $(\mathrm{BSB})_1$ and $(\mathrm{BSB})_2$ hold, $(N_k)_{k=-n}^{m}$ is indeed a basis of $\mathbb{S}$.

There exists an essential link between blossoms and B-spline bases, as recalled in the following:

Theorem 3.2. ([5], Theorem 3.3) *The following two statements are equivalent*:

(1) *blossoms exist in* $\mathbb{S}$;
(2) *there exists a B-spline basis in* $\mathbb{S}$ *and in any spline space derived from* $\mathbb{S}$ *by knot insertion.*

Its implication (1) $\Rightarrow$ (2) results from the properties of blossoms: indeed, they enable us to evaluate all values of Σ as convex combinations of the poles Π_i of Σ via a de Boor-type algorithm, namely

$$\Sigma(x) = \sum_{i=-n}^{m} N_i(x)\Pi_i, \quad \text{where } \Pi_i = \sigma(\xi_{i+1}, \dots, \xi_{i+n}) \in \mathbb{R}^{n+m}, \ \sum_{i=-n}^{m} N_i(x) = 1.$$

The sequence $(N_k)_{k=-n}^{m}$ of scalar coefficients forms a B-spline basis of $\mathbb{S}$. Also due to the properties of blossoms is the following feature of the B-spline bases obtained in this way: whenever blossoms exist in the spline space $\mathbb{S}$, the B-spline bases of $\mathbb{S}$ and of spaces obtained from it by knot insertion satisfy appropriate decomposition relations. Namely, when knot insertion is considered, blossoms produce a relation between old and new poles, which, in turn, yields a relation between old and new B-spline bases with the same support properties for the resulting coefficients (discrete B-splines) as in the standard polynomial spline case.

In the space $D\mathbb{S}$ we cannot expect to have a B-spline basis as defined above, since $D\mathbb{S}$ does not have to contain constants. Instead we make the following:

Definition 3.3 *(quasi-B-spline basis).* We say that a sequence $(B_k)_{k=-n+1}^{m}$ of splines in $D\mathbb{S}$ is a *quasi-B-spline basis of* $D\mathbb{S}$ if it satisfies properties $(\mathrm{BSB})_1$ and $(\mathrm{BSB})_2$ (which are as in Definition 3.1 but with $n = \dim D\mathbb{E}_k$ substituted everywhere in place of $n + 1 = \dim \mathbb{E}_k$).

From a B-spline basis in $\mathbb{S}$ we can construct a quasi-B-spline basis in $D\mathbb{S}$; furthermore, decomposition relations in $\mathbb{S}$ (relative to knot insertion) lead to appropriate decomposition relations in $D\mathbb{S}$.

The proof of implication (1) $\Rightarrow$ (2) of our main theorem relies on the existence of quasi-B-spline bases in derivative spline spaces and on the associated relevant decomposition relations. It is essentially modelled on the proof of

Theorems 2 and 3 in [2]. The proof of the reverse implication (2) $\Rightarrow$ (1) proceeds via the construction of quasi-B-spline bases in derivative spaces under the Schoenberg–Whitney conditions, then of B-spline bases in original spaces, finally via Theorem 3.2 to the existence of blossoms.

4. Comments

Whenever we know, from some other line of reasoning, that blossoms exist in $\mathbb{S}$, equivalence (1) $\Leftrightarrow$ (2) and implication (1) $\Rightarrow$ (3) automatically furnish us with Schoenberg–Whitney type theorems for the spline spaces $D\mathbb{S}$ and $\mathbb{S}$ (and all spline spaces obtained from them as specified in the theorem).

1) When considering spline spaces, either polynomial or Chebyshevian, with connection matrices, it has become conventional to impose the requirement of *total positivity* on these matrices. For polynomial splines this assumption applies to matrices connecting ordinary derivative vectors; whereas for Chebyshevian splines it applies to matrices connecting generalised derivative vectors, defined by means of *weight functions* (see [1]). There are, however, certain basic inconveniences inherent in the use of weight functions and of connection matrices expressed with their help:

– their choice is not unique: for a given EC-space of dimension greater than one, there always exists an *infinity* of ways to choose essentially different weight functions;
– a connection matrix which is totally positive with one choice of weight functions for the two adjacent EC section spaces, may fail to be totally positive with a different choice of weight functions.

The characteristic furnished by our theorem is free from these disadvantages: it is formulated in terms of blossoms (and of B-spline bases, through Theorem 3.2), whose existence is an intrinsic property of a spline space and does not depend on an arbitrary choice of weight functions.

2) It is known (see [3], Theorem 6.10) that total positivity of connection matrices guarantees the existence of blossoms. We thus obtain the following:

Corollary 4.1. *If all connection matrices N_k (linking generalised derivative vectors) of a spline space $\mathbb{S}$ are totally positive, then in $\mathbb{S}$, in $D\mathbb{S}$, as well as in any spline space obtained from either of them by a combination of knot insertion and restriction to a subinterval, an Hermite interpolation problem has a unique solution if and only if it satisfies the Schoenberg–Whitney conditions.*

The assertion about $\mathbb{S}$ and spaces obtained from it recovers Theorems 4.1 and 4.2 of [7], while the assertion about $D\mathbb{S}$ and spaces obtained from it is new.

The use of blossoms also empowers one to obtain new results by venturing into the largely unexplored ground beyond total positivity. Thus, in [4] the second author has explored the case $n = 3$ of EC-splines with connection matrices of order two (the dimensional equivalent of cubic splines) and has derived conditions on the connection matrices M_k equivalent to the existence of blossoms. It has thus been shown that blossoms can exist even when the connection matrices N_k fail to be totally positive: therefore, the requirement of existence of blossoms is *strictly weaker* than the requirement of total positivity.

The use of blossoms thus supplies two advantages over the standard setting, both of which are novel with respect to the existing literature on the subject. An article containing all the proofs of this Note is presently in preparation.

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