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On the essential spectrum of magnetic pseudodifferential operators

Marius Măntoiu [a], Radu Purice [a], Serge Richard [b]

[a] *Institute of Mathematics Simion Stoilow of the Romanian Academy, P.O. Box 1-764, Bucharest, RO-014700, Romania*
[b] *Université de Lyon, Lyon, F-69003, France; Université Lyon 1, Institut Camille Jordan, Villeurbanne Cedex, F-69622, France; CNRS, UMR 5208, Villeurbanne Cedex, F-69622, France*

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Abstract

We study magnetic pseudodifferential operators associated with elliptic symbols and with anisotropic potentials. We prove affiliation to suitable C^*-algebras and give formulae for the essential spectrum as a union of spectra of some asymptotic operators. ***To cite this article: M. Măntoiu et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

Résumé

Sur le spectre essentiel des opérateurs pseudodifférentiels magnétiques. Nous étudions des opérateurs pseudodifférentiels magnétiques associés à des symboles elliptiques et ayant des potentiels anisotropes. Nous démontrons leur affiliation à certaines C^*-algèbres et nous donnons des formules pour le spectre essentiel comme une union des spectres de certains opérateurs asymptotiques. ***Pour citer cet article : M. Măntoiu et al., C. R. Acad. Sci. Paris, Ser. I 344 (2007).***

1. Introduction

In [8] we have defined a gauge-covariant quantization for a particle in a magnetic field, extending the Weyl pseudodifferential calculus (see also [6]). Let X be $\mathbb{R}^N$, $X^\star$ its dual, $\Xi := X \times X^\star$ and $\mathcal{H} := L^2(X)$. For a continuous vector potential A generating a continuous magnetic field B and for any $f : \Xi \to \mathbb{C}$ we define *the magnetic pseudodifferential operator*

$$\big[\mathfrak{Op}^A(f)u\big](x) := \int_X \mathrm{d}y \int_{X^\star} \mathrm{d}p\, \mathrm{e}^{\mathrm{i}p\cdot(x-y)} \lambda^A(x; y-x) f\left(\frac{1}{2}(x+y), p\right) u(y), \quad u \in \mathcal{H},$$

where $\lambda^A(q;x) := \exp(-\mathrm{i}\Gamma^A[q, q+x])$ and $\Gamma^A[q, q+x]$ is the circulation of A from q to $q+x$. *The magnetic Moyal product* acting on functions $f, g : \Xi \to \mathbb{C}$ verifies $\mathfrak{Op}^A(f \circ g) = \mathfrak{Op}^A(f)\mathfrak{Op}^A(g)$. This operation is defined for $\xi = (q, p)$, $\eta = (x, k)$ and $\zeta = (y, l)$ in Ξ by

E-mail addresses: marius.mantoiu@imar.ro (M. Măntoiu), radu.purice@imar.ro (R. Purice), srichard@math.univ-lyon1.fr (S. Richard).

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$$[f \circ g](\xi) := 4^N \int_\Xi \mathrm{d}\eta \int_\Xi \mathrm{d}\zeta\, \mathrm{e}^{-2\mathrm{i}(k\cdot y - l\cdot x)} \omega^B\big(q - x - y; 2x, 2(y-x)\big) f(\xi - \eta) g(\xi - \zeta), \tag{1}$$

where $\omega^B(q;x,y) := \exp(-\mathrm{i}\Gamma^B\langle q, q+x, q+x+y\rangle)$ and $\Gamma^B\langle q, q+x, q+x+y\rangle$ is the magnetic flux through the triangle defined by q, $q+x$ and $q+x+y$. These integrals are absolutely convergent only for restricted classes of symbols; for more general distributions we require that the components B_{jk} of the magnetic field belong to $C^\infty_{\mathrm{pol}}(X)$, i.e. they are indefinitely derivable and each derivative is polynomially bounded. Under this assumption, extending (1) by duality, we define *the magnetic Moyal* **-algebra* $\mathcal{M}(\Xi)$, which contains the space $C^\infty_{\mathrm{pol,u}}(\Xi)$ of functions on Ξ with uniform polynomial growth at infinity. $\mathfrak{Op}^A$ extends to $\mathcal{M}(\Xi)$, as continuous operators, resp. in the Schwartz space $\mathcal{S}(X)$ and in its dual $\mathcal{S}'(X)$.

In [9] we have introduced a related C^*-algebraic framework. Let $BC_u(X)$, resp. $C_0(X)$, denote the algebra of bounded, uniformly continuous functions on X, resp. the ideal of continuous functions on X vanishing at infinity. We consider a unital C^*-subalgebra $\mathcal{A}$ of $BC_u(X)$ (encoding the anisotropy) containing $C_0(X)$ and stable by translations, i.e. $\theta_x(a) := a(\cdot + x) \in \mathcal{A}, \forall a \in \mathcal{A}, x \in X$. Assuming that $B_{jk} \in \mathcal{A}$, the map $X \times X \ni (x,y) \mapsto \omega^B(x,y) := \omega^B(\cdot; x, y)$ is a 2-cocycle on X with values in the unitary group $\mathcal{U}(\mathcal{A})$ of $\mathcal{A}$. We define the product on $L^1(X;\mathcal{A})$:

$$(\phi \diamond \psi)(x) := \int_X \mathrm{d}y\, \theta_{\frac{y-x}{2}}\big[\phi(y)\big]\theta_{\frac{y}{2}}\big[\psi(x-y)\big]\theta_{-\frac{x}{2}}\big[\omega^B(y, x-y)\big], \quad \phi, \psi \in L^1(X;\mathcal{A})$$

and the involution $\phi^\diamond(x) := \phi(-x)^*$. The associated C^*-algebra is called *the twisted crossed product* and is denoted by $\mathcal{A} \rtimes^\omega_\theta X$ (or $\mathfrak{C}^B_{\mathcal{A}}$ for shortness). Choosing a continuous vector potential A that generates B, one constructs a faithful, irreducible representation in $\mathcal{H}$ of the algebra $\mathfrak{C}^B_{\mathcal{A}}$:

$$\big[\mathfrak{Rep}^A(\phi)u\big](x) = \int_X \mathrm{d}y\, \lambda^A(x; y-x)\phi\bigg(\frac{1}{2}(x+y); y-x\bigg)u(y), \quad \phi \in L^1(X;\mathcal{A}),\ u \in \mathcal{H}.$$

$\mathfrak{Rep}^A$ and $\mathfrak{Op}^A$ are connected by a partial Fourier transformation $\mathfrak{F}$. The enveloping C^*-algebra $\mathfrak{B}^B_{\mathcal{A}}$ of $\mathfrak{F}(L^1(X;\mathcal{A}))$, endowed with the multiplication $\circ$ and with the complex conjugation, is isomorphic through this partial Fourier transformation to $\mathfrak{C}^B_{\mathcal{A}}$ and one has $\mathfrak{Op}^A(\mathfrak{B}^B_{\mathcal{A}}) = \mathfrak{Rep}^A(\mathfrak{C}^B_{\mathcal{A}})$.

2. Results

Definition 2.1. (1) An observable affiliated to a C^*-algebra $\mathfrak{C}$ is a morphism $\Phi : C_0(\mathbb{R}) \to \mathfrak{C}$.

(2) A function $h \in C^\infty(X^\star)$ is *a symbol of type s* if $\forall \alpha \in \mathbb{N}^N$, $\exists c_\alpha > 0$ such that $|(\partial^\alpha h)(p)| \leqslant c_\alpha \langle p\rangle^{s-|\alpha|}$ for all $p \in X^\star$, where $\langle p\rangle := \sqrt{1+p^2}$. In this case, we write $h \in S^s(X^\star)$.

(3) For $s > 0$, the symbol h is called *elliptic* if there exist $R, c > 0$ such that $c\langle p\rangle^s \leqslant h(p)$ for all $p \in X^\star$ and $|p| \geqslant R$. We denote by $S^s_{\mathrm{el}}(X^\star)$ the family of elliptic symbols of type s, and set $S^\infty_{\mathrm{el}}(X^\star) := \bigcup_s S^s_{\mathrm{el}}(X^\star)$.

The class $S^s(X^\star)$ is contained in $C^\infty_{\mathrm{pol,u}}(\Xi) \subset \mathcal{M}(\Xi)$. For $z \notin \mathbb{R}$, we define $r_z : \mathbb{R} \to \mathbb{C}$, $r_z(t) := (t-z)^{-1}$. $BC^\infty(X)$ denote the space of complex functions on X with bounded derivatives of any order.

Hypothesis 2.2. B is a magnetic field with components in $\mathcal{A} \cap BC^\infty(X)$ and V is a real element of $\mathcal{A}$.

Theorem 2.3. *Under Hypothesis* 2.2, *each real* $h \in S^\infty_{\mathrm{el}}(X^\star)$ *defines an observable* $\Phi^B_{h,V}$ *affiliated to* $\mathfrak{B}^B_{\mathcal{A}}$, *such that for* $z \notin \mathbb{R}$

$$(h + V - z) \circ \Phi^B_{h,V}(r_z) = 1 = \Phi^B_{h,V}(r_z) \circ (h + V - z).$$

Corollary 2.4. *In the framework of Theorem* 2.3 *let* A *be a continuous vector potential generating* B. *Then* $\mathfrak{Op}^A(h) + V$ *defines a selfadjoint operator* $H_h(A,V)$ *in* $\mathcal{H}$ *with domain equal to the range of the operator* $\mathfrak{Op}^A[(h-z)^{-1}]$ (*not depending on* $z \notin \mathbb{R}$). *This operator is affiliated to* $\mathfrak{Op}^A(\mathfrak{B}^B_{\mathcal{A}})$.

Theorem 2.3 leads to a decomposition of the essential spectrum of $H_h(A,V)$ prescribed by the behaviour at infinity of B and V. The aims and techniques of proving this result are in a certain relationship with those of [1–5,7] and [11]. Detailed proofs and examples are given in [10].

Let $S_{\mathcal{A}}$ be the spectrum of $\mathcal{A}$; $X \subset S_{\mathcal{A}}$ open and dense. $\mathcal{A}$ being stable under translations, the group law $\theta : X \times X \to X$ extends to a continuous map $\tilde{\theta} : X \times S_{\mathcal{A}} \to S_{\mathcal{A}}$. We denote by $F_{\mathcal{A}}$ the complement of X in $S_{\mathcal{A}}$. To any point of $F_{\mathcal{A}}$ we associate its quasi-orbit (the closure of its orbit under $\tilde{\theta}$). Given any quasi-orbit F we can define $\widetilde{V_F} \in C(F)$ as the restriction of $V \in \mathcal{A} \equiv C(S_{\mathcal{A}})$ to F (it is in fact defined as limit at infinity along the ultrafilters that belong to F). Similarly we can proceed with the components of the magnetic field and define its restriction $\widetilde{B_F}$. Once we fix an element $\mathfrak{x} \in F$ we can associate to any element $\tilde{F} \in C(F)$ an element $F \in BC_u(X)$ defined by $F(x) := \tilde{F}(\tilde{\theta}(x,\mathfrak{x}))$. We obtain in this way V_F and B_F; A, A_F denote then continuous vector potentials for B and B_F. Let us now consider a covering of $F_{\mathcal{A}}$ by quasi-orbits $\{F_\nu\}_\nu$.

Theorem 2.5. *Under Hypothesis* 2.2 *and using the above construction, for each real* $h \in S^\infty_{\mathrm{el}}(X^\star)$ *we have*:

$$\sigma_{\mathrm{ess}}\big[H_h(A,V)\big] = \overline{\bigcup_\nu \sigma\big[H_h(A_\nu,V_\nu)\big]},$$

where $B_\nu \equiv B_{F_\nu}$ *and* $V_\nu \equiv V_{F_\nu}$.

The localization results proved in [2] (where their physical interpretation is discussed) extend to our situation. For a quasi-orbit F, let $\mathcal{N}_F$ be a base of neighbourhoods of F in $S_{\mathcal{A}}$, $W := \mathcal{W} \cap X$ for any $\mathcal{W} \in \mathcal{N}_F$ and let $\chi_W(Q)$ denote the multiplication operator with the characteristic function on W.

Theorem 2.6. *Under Hypothesis* 2.2 *let h be a real element of* $S^\infty_{\mathrm{el}}(X^\star)$. *Assume that F is a quasi-orbit and let A, A_F be continuous vector potentials for B and B_F. If* $\eta \in C_0(\mathbb{R})$ *with* $\mathrm{supp}(\eta) \cap \sigma[H_h(A_F,V_F)] = \emptyset$ *(an energy cut-off outside the spectrum of* $H_h(A_F,V_F)$*), then for any* $\varepsilon > 0$ *there exists* $\mathcal{W} \in \mathcal{N}_F$ *such that* $\|\chi_W(Q)\eta[H_h(A,V)]\| \leqslant \varepsilon$. *In particular, the inequality* $\|\chi_W(Q)\mathrm{e}^{-\mathrm{i}tH_h(A,V)}\eta[H_h(A,V)]u\| \leqslant \varepsilon\|u\|$ *holds, uniformly in* $t \in \mathbb{R}$ *and* $u \in \mathcal{H}$.

3. Sketch of the Proof of Theorem 2.3

Let $(\mathcal{M},\circ)$ be an associative algebra. We look for an inverse of $\mathfrak{h} \in \mathcal{M}$. Suppose that there exists $\mathfrak{h}' \in \mathcal{M}$ such that $\mathfrak{h} \circ \mathfrak{h}'$ and $\mathfrak{h}' \circ \mathfrak{h}$ have inverses $(\mathfrak{h} \circ \mathfrak{h}')^{(-1)}$ and $(\mathfrak{h}' \circ \mathfrak{h})^{(-1)}$. Then $\mathfrak{h}' \circ (\mathfrak{h} \circ \mathfrak{h}')^{(-1)}$ is obviously a right inverse for $\mathfrak{h}$ and $(\mathfrak{h}' \circ \mathfrak{h})^{(-1)} \circ \mathfrak{h}'$ a left inverse for $\mathfrak{h}$. Both are thus equal to $\mathfrak{h}^{(-1)}$. We shall take for $\mathfrak{h}$ the strictly positive symbol $h + a$, with a large enough, and for $\mathfrak{h}'$ its pointwise inverse $(h+a)^{-1}$. In the complete proof several arguments need regularizations.

We consider an elliptic symbol h of order s, fix $a \geqslant -\inf h + 1$, set $h_a := h + a$, and denote by h_a^{-1} its inverse for pointwise multiplication (a symbol of type $-s$). Since h_a, h_a^{-1} are in $C^\infty_{\mathrm{pol,u}}(\Xi)$:

$$\big(h_a \circ h_a^{-1}\big)(q,p) = 4^N \int_X \mathrm{d}x \int_{X^\star} \mathrm{d}k \int_X \mathrm{d}y \int_{X^\star} \mathrm{d}l\, \mathrm{e}^{-2\mathrm{i}(k\cdot y - l\cdot x)} \gamma^B(q;2x,2y) \frac{h_a(p-k)}{h_a(p-l)},$$

where $\gamma^B(q;2x,2y) := \omega^B(q-x-y;2x,2(y-x))$. The last factor has a Taylor expansion:

$$\frac{h_a(p-k)}{h_a(p-l)} = 1 + \sum_{j=1}^N (l_j - k_j) \frac{\int_0^1 \mathrm{d}t\, (\partial_j h)(p - l + t(l-k))}{h(p-l)+a} =: 1 + \sum_{j=1}^N F_{a,j}(p;k,l).$$

Denote $\langle\cdot,\cdot\rangle$, the duality between $C^\infty_{\mathrm{pol,u}}(X^\star \times X^\star)$ and the Fourier transform $\mathbb{F}C^\infty_{\mathrm{pol}}(X^\star \times X^\star)$ and obtain estimates for $f_{a,j}(q;p) := \langle(\mathbb{F}\gamma^B)(q;\cdot,\cdot), F_{a,j}(p;\cdot,\cdot)\rangle$. Our hypothesis on h and B imply that for any $\mu > \max\{1,s\}$ and any multi-index $\alpha \in \mathbb{N}^N$: $|(\partial_p^\alpha f_{a,j})(q;p)| \leqslant c a^{-1/\mu}\langle p\rangle^{s/\mu - 1 - |\alpha|}$, where c depends on α and j but not on p, q or a [10]. It is easy to prove that $f_{a,j}(\cdot;p)$ belongs to $\mathcal{A}$, for all $p \in X^\star$. Then as in [1, Proposition 1.3.3] and [1, Proposition 1.3.6] one obtains the estimate $\|\mathfrak{F}^{-1}(f_{a,j})\|_1 \leqslant C a^{-1/\mu}$. Thus, for a large enough, $\|\sum_{j=1}^N \mathfrak{F}^{-1}(f_{a,j})\|_1 < 1$ holds. It follows that $\mathfrak{F}^{-1}(1 + \sum_{j=1}^N f_{a,j})$ is invertible in the minimal unitization of $L^1(X;\mathcal{A})$. Equivalently, $h_a \circ h_a^{-1} \equiv 1 + \sum_{j=1}^N f_{a,j}$ is invertible in the minimal unitization of $\mathfrak{F}(L^1(X;\mathcal{A}))$. Its inverse will be denoted by $(h_a \circ h_a^{-1})^{(-1)}$. By the same arguments (see [1, Proposition 1.3.6]) we get $h_a^{-1} \in \mathfrak{F}(L^1(X)) \subset \mathfrak{F}(L^1(X;\mathcal{A}))$. Thus $h_a^{-1} \circ (h_a \circ h_a^{-1})^{(-1)}$ is a well defined element of $\mathfrak{F}(L^1(X;\mathcal{A}))$. Moreover, one readily gets $h_a \circ [h_a^{-1} \circ (h_a \circ h_a^{-1})^{(-1)}] = 1$. In the same way one obtains $[(h_a^{-1} \circ h_a)^{(-1)} \circ h_a^{-1}] \circ h_a = 1$ in $\mathcal{M}(\Xi)$. In conclusion, there exists $a_0 \geqslant -\inf h + 1$ such that for any $a > a_0$ the symbol h_a has an inverse $h_a^{(-1)} \in \mathfrak{F}(L^1(X;\mathcal{A})) \subset \mathfrak{B}_{\mathcal{A}}^B$. We define $\Phi_h^B(r_x) := h_{-x}^{(-1)}$ for $x < -a_0$. Then $\Phi_h^B(r_x) \in \mathfrak{F}(L^1(X;\mathcal{A})) \subset \mathfrak{B}_{\mathcal{A}}^B \cap \mathcal{S}'(\Xi)$, its norm is uniformly bounded for x in the given domain and $(h-x) \circ \Phi_h^B(r_x) = \Phi_h^B(r_x) \circ (h-x) = 1$, as shown above. This allows us to obtain an extension to the half-strip $\{z = x + \mathrm{i}y \mid x < -a_0,\ |y| < \delta\}$ for some $\delta > 0$. We end the proof by verifying the resolvent equation for the map

$$\{z = x + \mathrm{i}y \mid x < -a_0,\ |y| < \delta\} \ni z \mapsto \Phi_h^B(r_z) \in \mathfrak{F}\big(L^1(X;\mathcal{A})\big).$$

A general argument presented in [1, p. 364] allows now to extend the map Φ_h^B to a C^*-algebra morphism $C_0(\mathbb{R}) \to \mathfrak{B}_{\mathcal{A}}^B$. The observable $\Phi_{h,V}^B$ is finally obtained by a perturbative argument [10].

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