Probability Theory

On the longest common increasing binary subsequence

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Abstract

Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be two independent sequences of iid Bernoulli random variables with parameter $1/2$. Let $LCI_n$ be the length of the longest increasing sequence which is a subsequence of both finite sequences $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$. We prove that, as $n$ goes to infinity, $n^{-1/2}(LCI_n - n/2)$ converges in law to a Brownian functional that we identify. To cite this article: C. Houdré et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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Résumé

Sur la plus longue sous-suite binaire croissante commune. Soient $X_1, X_2, \ldots$ et $Y_1, Y_2, \ldots$ deux suites mutuellement indépendantes de variables aléatoires de Bernoulli indépendantes, équidistribuées de paramètre $1/2$. Soit $LCI_n$ la longueur de la plus longue sous-suite croissante et commune aux deux sous-suites finies $X_1, \ldots, X_n$ et $Y_1, \ldots, Y_n$. Nous démontrons que, lorsque $n$ tend vers l’infini, $n^{-1/2}(LCI_n - n/2)$ converge en loi vers une fonctionnelle brownienne que nous identifions. Pour citer cet article : C. Houdré et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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Version française abrégée

Dans cette Note, nous nous intéressons au problème de la détermination de l’ordre des fluctuations de la longueur de la plus longue sous-suite croissante qui est aussi commune à deux suites binaires indépendantes. Notre résultat principal est le suivant :

Théorème. Soient $X_1, X_2, \ldots, X_n$ et $Y_1, Y_2, \ldots, Y_n$ deux suites indépendantes de variables aléatoires de Bernoulli iid et de paramètre $1/2$. Soit $LCI_n$ le maximum des entiers $1 \leq k \leq n$ tels qu’il existe $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ et $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ avec $X_{i_1} \leq X_{i_2} \leq \cdots \leq X_{i_k}$, $Y_{j_1} \leq Y_{j_2} \leq \cdots \leq Y_{j_k}$ et $X_{i_s} = Y_{j_s}$ pour tout $s = 1, \ldots, k$. Alors,

$$\frac{LCI_n - n/2}{\sqrt{n}} \to \max_{t \in [0,1]} \min_{i=1,2} \left( B^i(t) - \frac{1}{2} B^i(1) \right),$$

où $B^1 = (B^1(t))_{t \in [0,1]}$ et $B^2 = (B^2(t))_{t \in [0,1]}$ sont deux mouvements browniens standards indépendants.

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Les motivations et perspectives de notre travail sont à la fois les interactions profondes entre les problèmes de sous-suites et diverses branches des mathématiques comme la combinatoire algébrique, les matrices aléatoires, et les polynômes orthogonaux, mais aussi des considérations plus pratiques de bioinformatique. L’obtention de lois limites, est en effet une première étape vers le développement de méthodes quantitatives, telles que des tests statistiques, utilisables pour le séquençage de l’ADN.

1. Introduction: the one sequence case

Longest increasing subsequence (LIS) problems have recently enjoyed renewed popularity. This stems mainly from the work of Baik, Deift and Johansson [1] who showed that the limiting law of the fluctuations of the longest increasing subsequence of a random permutation is the same as the law of the maximal eigenvalue of certain random matrix models. This result has led to numerous recent advances, such as the work of Borodin [3] for colored random permutations or the case of finite alphabets random words by Its, Tracy and Widom [8,4,5] as well as Johansson [6]. Related results with links to queueing theory were also obtained by Baryshnikov [2] and further connections with orthogonal polynomials, random matrices, growth models have since been further investigated by many authors.

A related important open problem, with direct consequences in computational biology (see, e.g. Waterman [9,10]), is to find the order of the fluctuations of the length of the longest common subsequence (LCS) of two or more random sequences.

We obtain below the limiting distribution for the hybrid problem of the longest common and increasing subsequence (LCIS) of two random binary sequences. We start by presenting the one sequence case, where the results are known and obtained in the works just cited. Our approach might be worthwhile because of its simplicity and because it naturally leads and extends to the case of two (or more) sequences.

Let \( X := (X_1, X_2, \ldots) \in \{0,1\}^\mathbb{N} \) be an infinite binary sequence. Let \( LI_n \) be the length of the longest increasing subsequence of \( X_1, X_2, \ldots, X_n \), i.e. \( LI_n \) is the maximal \( k \leq n \) such that there exists an increasing sequence of natural numbers \( 1 \leq i_1 < i_2 < \cdots < i_k \leq n \) such that \( X_{i_1} \leq X_{i_2} \leq \cdots \leq X_{i_k} \). Let \( b_k \) be the number of ones in the finite sequence \( X_1, X_2, \ldots, X_k \), in other words, let \( b_0 := 0 \), \( b_k := \sum_{i=1}^k X_i \), and let also \( a_k \) be the number of zeros in the sequence \( X_1, X_2, \ldots, X_k \). Clearly, \( a_k = k - b_k \). Next, for every \( 0 \leq k \leq n \), an increasing subsequence of \( X_1, X_2, \ldots, X_n \) can be constructed by taking all the zeros up to (including) \( X_k \), and then by taking all the ones between (and including) \( X_{k+1} \) and \( X_n \). The number of zeros up to time \( k \) is equal to \( a_k \), while the number of ones from \( X_k \) to \( X_n \) is equal to \( b_n - b_k \). The maximum over \( k = 0, \ldots, n \) of the length of all the subsequences obtained in this way is \( LI_n \). In other words,

\[
LI_n = \max_{k=0,\ldots,n} (a_k + (b_n - b_k)) = b_n + \max_{k=0,\ldots,n} (k - 2b_k).
\]

Letting \( Z_i := 1 - 2X_i \), it is then clear that \( a_k - b_k = k - 2b_k = \sum_{i=1}^k Z_i \), and so setting \( S_0 = 0 \), \( S_k = \sum_{i=1}^k Z_i \), \( k \geq 1 \), gives:

\[
LI_n = \frac{n}{2} - \frac{S_n}{2} + \max_{k=0,\ldots,n} S_k. \tag{1}
\]

Using (1) and the reflection principle, one easily shows that \( \mathbb{E} LI_n = n/2 + \sqrt{2n/\pi} + o(\sqrt{n}) \) and that \( \text{Var} \ LI_n = 3n/4 - 2n/\pi + o(n) \). Moreover, Donsker’s theorem and the continuous mapping theorem yield

\[
\frac{LI_n - n/2}{\sqrt{n}} \xrightarrow{c} - \frac{B(1)}{2} + \max_{t \in [0,1]} B(t), \tag{2}
\]

where ‘\( \xrightarrow{c} \)’ stands for convergence in law. The above limiting law is well known and connected to a theorem of Pitman (see [7]). Its density can also be derived ‘by hand’ and is given by: \( 16x^2e^{-2x^2}/\sqrt{2\pi}, \ x > 0 \).

2. The two sequence case

Let \( X_1, X_2, \ldots \) and \( Y_1, Y_2, \ldots \) be binary sequences, and let,

\[
X^n := (X_1, \ldots, X_n), \quad Y^n := (Y_1, \ldots, Y_n).
\]
Denote by $LCI_n$ the length of the longest common increasing subsequence which is contained in both $X^n$ and $Y^n$. In other words, $LCI_n$ is the maximum over the $k$s that satisfy the following conditions: there exist $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ and $1 \leq j_1 < j_2 < \cdots < j_k \leq n$ such that

$$X_{i_1} \leq X_{i_2} \leq \cdots \leq X_{i_k}, \quad Y_{j_1} \leq Y_{j_2} \leq \cdots \leq Y_{j_k},$$

and $X_{i_s} = Y_{j_s}$ for all $s = 1, \ldots, k$. Let $N_1$ (resp. $N_2$) be the number of zeros in $X^n$ (resp. in $Y^n$). Let $T^1_k$ denote the location of the $k$th zero in the sequence $(X_1, X_2, \ldots)$, i.e. $T^1_k$ is defined recursively by the equations:

$$T^1_0 = 0, \quad T^1_1 = \min\{t: X_t = 0\}, \quad T^1_{k+1} = \min\{t > T^1_k: X_t = 0\}.$$ 

In a similar way define $T^2_k$ to be the location of the $k$th zero in the sequence $(Y_1, Y_2, \ldots)$. Let,

$$g^1: \{0, \ldots, N_1\} \to \mathbb{N} \quad \text{(resp. } g^2: \{0, \ldots, N_2\} \to \mathbb{N})$$

be the maximum number of ones contained in any increasing subsequence of $X^n$ (resp. of $Y^n$) which contains exactly $k$ zeros. Hence, $g^1(k) = \sum_{i>T^1_k} X_i$, $k = 0, \ldots, N_1$, $g^2(k) = \sum_{i>T^2_k} Y_i$, $k = 0, \ldots, N_2$, and, in particular, $g^1(0) = n - N_1$. Thus, $g^1(k) + k$ (resp. $g^2(k) + k$) is the length of the longest increasing subsequence of $X^n$ (resp. $Y^n$) that contains exactly $k$ zeros and $\min_{i=1,2} g^i(k) + k$ is the length of the longest common increasing subsequence with exactly $k$ zeros. Since a common subsequence of $X$ and $Y$ can contain at most $N_1 \wedge N_2$ zeros, we have the following useful relation:

$$LCI_n = \max_{k=0, \ldots, n, N_1, N_2} \left[ \min_{i=1,2} \left( g^i(k) + k \right) \right]. \quad (3)$$

Let $1 \leq k \leq N_i$. Between $k = 1$ and $k$, the function $g^i$ decreases by the number of ones located between $T^i_{k-1}$ and $T^i_k$. This number is equal to $Z^i_k := T^i_k - T^i_{k-1} - 1$, $k = 1, \ldots, N_i$. Thus, for $i = 1, 2$, it follows that

$$g^i(k) - g^i(k-1) = -Z^i_k, \quad k = 1, \ldots, N_i. \quad (4)$$

Moreover, recall that $g^i(0) = n - N_i$, and thus for any $k \geq 1$,

$$g^i(k) = n - N_i - \sum_{j=1}^k Z^i_j, \quad i = 1, 2. \quad (5)$$

Assume now that the sequences $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ are independent of each other. Let also the $X_k$s as well as the $Y_k$s be iid Bernoulli random variables with parameter $1/2$. In this case, $T^1_1, T^1_2, \ldots, T^1_k, \ldots$ are Pascal (negative binomial) random variables with respective parameters $1, 2, \ldots, k, \ldots$ and $1/2$ and, as such each $T^1_k$ is the sum of $k$ iid geometric random variables with parameter $1/2$. Now, for $i = 1, 2, Z^1_i + 1, Z^2_i + 1, Z^1_i + 1, \ldots$ is the corresponding sequence of iid geometric random variables with parameter $1/2$. Hence $Z^1_1, Z^1_2, Z^1_3, \ldots$ is a sequence of iid random variables with $E(Z^1_i) = 1$ and $\text{Var} Z^1_i = 2$. Moreover the sequences $Z^1_1, Z^1_2, Z^1_3, \ldots$ and $Z^2_1, Z^2_2, Z^2_3, \ldots$ are also independent. We use the sequence $Z^1_1, Z^1_2, Z^1_3, \ldots$ to approximate a standard Brownian motion. Let $k = 0, \ldots, n$ and $t = k/n$, and let $\hat{B}^i(t) := -\sum_{j=1}^t (Z^i_j - 1)/\sqrt{2n}$. For $t \in (k/n, (k+1)/n)$, $k = 0, 1, \ldots, n - 1$, define $\hat{B}^i(t)$ by linear interpolation. By (4) and (5), it thus follows that $g^i(k) + k = \hat{B}^i(0) - \sum_{j=1}^k (Z^i_j - 1) = n - N_i + \sqrt{2n} \hat{B}^i(k/n)$.

Hence, by (3),

$$LCI_n = \max_{0 \leq k \leq n, N_1, N_2} \left[ \left( n - N_1 + \sqrt{2n} \hat{B}^1_n \left( \frac{k}{n} \right) \right) \wedge \left( n - N_2 + \sqrt{2n} \hat{B}^2_n \left( \frac{k}{n} \right) \right) \right]. \quad (6)$$

Note that

$$T^i_k = \sum_{j=1}^k (Z^i_j + 1) = \sum_{j=1}^k (Z^i_j - 1) + 2k = -\sqrt{2n} \hat{B}^i_n \left( \frac{k}{n} \right) + 2k. \quad (7)$$

Moreover, $N_i$ is a binomial random variable with parameters $n$ and $1/2$, and thus for $n$ large it is highly concentrated around its mean $n/2$. 
The next theorem states that in this setting an analogue of (2) hold, i.e. \( n^{-1/2}(LCI_n - n/2) \) converges in law to a Brownian functional. This implies that \( LI_n \) as well as the stochastically smaller random variable \( LCI_n \) have the same order of fluctuation. Intuitively, to better understand this result, note that it is very likely that, in both sequences, the first \( n/2 \) terms contain around \( n/4 \) zeroes, and the last \( n/2 \) terms contain around \( n/4 \) ones. When this is the case, the longest common subsequence is at least \( n/2 \). However, since by (2), \( LI_n \) is equivalent to \( n/2 \), the stochastically smaller random variable \( LCI_n \) should have the same property.

**Theorem 2.1.** Let \( X_1, X_2, \ldots, X_n, \ldots \) and \( Y_1, Y_2, \ldots, Y_n, \ldots \) be two independent sequences of iid Bernoulli random variables with parameter 1/2. Then

\[
\frac{LCI_n - n/2}{\sqrt{n}} \Rightarrow \max_{t \in [0,1]} \left[ \min_{i=1,2} \left( B^i(t) - \frac{1}{2} B^i(1) \right) \right],
\]

where \( B^1 = (B^1(t))_{t \in [0,1]} \) and \( B^2 = (B^2(t))_{t \in [0,1]} \) are two independent standard Brownian motions.

**Proof.** By the self-similarity property of Brownian motion, to prove (8) it suffices to show that

\[
\frac{LCI_n - n/2}{\sqrt{2n}} \Rightarrow \max_{t \in [0,1]} \left[ \min_{i=1,2} \left( B^i(t) - \frac{1}{2} B^i(1/2) \right) \right].
\]

Next, for \( a_k, b_k \) reals,

\[
\left| \max_{k=1,\ldots,n} (a_k + b_k) - \max_{k=1,\ldots,n} (a_k + c) \right| \leq \max_{k=1,\ldots,n} \left| (a_k + b_k) - ((a_k + c) \wedge (b_k + d)) \right| \leq |c| \vee |d|.
\]

By (6),

\[
D_n := \frac{LCI_n - n/2}{\sqrt{2n}} = \max_{0 \leq k \leq N_1 \wedge N_2} \left[ \left( n/2 - N_1 \frac{k}{n} \right) \wedge \left( \frac{N_2}{n} \right) \right].
\]

Let

\[
\gamma_i := \frac{n/2 - N_i}{\sqrt{2n}} + \frac{1}{2} \hat{B}^i(n) \quad i = 1, 2.
\]

So

\[
D_n = \max_{0 \leq k \leq N_1 \wedge N_2} \left[ \left( \gamma_1^i - \frac{1}{2} \hat{B}^1 \left( \frac{N_1}{n} \right) + \hat{B}^1 \left( \frac{k}{n} \right) \right) \wedge \left( \gamma_2^i - \frac{1}{2} \hat{B}^2 \left( \frac{N_2}{n} \right) + \hat{B}^2 \left( \frac{k}{n} \right) \right) \right].
\]

Let

\[
U_n := \max_{0 \leq k \leq N_1 \wedge N_2} \left[ \left( -\frac{1}{2} \hat{B}^1 \left( \frac{N_1}{n} \right) + \hat{B}^1 \left( \frac{k}{n} \right) \right) \wedge \left( -\frac{1}{2} \hat{B}^2 \left( \frac{N_2}{n} \right) + \hat{B}^2 \left( \frac{k}{n} \right) \right) \right].
\]

By (10), we have:

\[
|D_n - U_n| \leq \left| \gamma_1^i \right| \vee \left| \gamma_2^i \right| .
\]

Let

\[
V_n := \max_{0 \leq k \leq N_1 \wedge N_2} \left[ \left( -\frac{1}{2} \hat{B}^1 \left( \frac{1}{2} \right) + \hat{B}^1 \left( \frac{k}{n} \right) \right) \wedge \left( -\frac{1}{2} \hat{B}^2 \left( \frac{1}{2} \right) + \hat{B}^2 \left( \frac{k}{n} \right) \right) \right].
\]

By (10), we have:

\[
|U_n - V_n| \leq \frac{1}{2} \left| \hat{B}^1 \left( \frac{1}{2} \right) - \hat{B}^1 \left( \frac{N_1}{n} \right) \right| \vee \frac{1}{2} \left| \hat{B}^2 \left( \frac{1}{2} \right) - \hat{B}^2 \left( \frac{N_2}{n} \right) \right| .
\]

Let

\[
X_n := \max_{0 \leq t \leq 1/2} \left[ \left( -\frac{1}{2} \hat{B}^1 \left( \frac{1}{2} \right) + \hat{B}^1 \left( t \right) \right) \wedge \left( -\frac{1}{2} \hat{B}^2 \left( \frac{1}{2} \right) + \hat{B}^2 \left( t \right) \right) \right].
\]
Hence,
\[ V_n - X_n \leq \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left( B_n^1(t) - \hat{B}_n^1 \left( \frac{1}{2} \right) \right) \vee \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left( B_n^2(t) - \hat{B}_n^2 \left( \frac{1}{2} \right) \right). \] (13)

In the following, let \( i = 1, 2 \) be fixed, and let us skip it from the notation. By the very definition of \( \hat{B}_n \),
\[ \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left( B_n(t) - \hat{B}_n \left( \frac{1}{2} \right) \right) = \max_{k = n/2, \ldots, N} \left( \hat{B}_n \left( \frac{k}{n} \right) - \hat{B}_n \left( \frac{1}{2} \right) \right) \vee 0. \]

Let \( m = \lceil n/2 \rceil \), where \( \lceil \cdot \rceil \) is the usual ceiling (or greatest integer) function. Then, \( \hat{B}_n(k/n) = \hat{B}_n(1/2) = \sum_{j=m}^{k} \xi_j / \sqrt{2n} \), where
\[ \xi_m = \sqrt{2n} \left( \hat{B}_n \left( \frac{m}{n} \right) - \hat{B}_n \left( \frac{1}{2} \right) \right), \quad \xi_{m+1} = Z_{m+1} - 1, \quad \xi_{m+2} = Z_{m+2} - 1, \ldots, \xi_k = Z_k - 1. \]

Clearly, \( \xi_m = 0 \), if \( n \) is even, and \( \xi_m = \frac{1}{2} (Z_m - 1) \) otherwise. Let \( C_n := \{ \lfloor N/n \rfloor = \frac{1}{2} \} \), then
\[ P \left( \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left| \hat{B}_n(t) - \hat{B}_n \left( \frac{1}{2} \right) \right| > \varepsilon \right) \leq P \left( \max_{k = m, \ldots, n} \frac{1}{\sqrt{2n}} \left| \sum_{j=m}^{k} \xi_j \right| > \varepsilon \right) + P(C_n) \leq \frac{\ln n}{\varepsilon^2 \sqrt{n}} + P(C_n), \]

by Kolmogorov’s inequality. Next, \( P(C_n) \to 0 \). Indeed, \( N \sim Bin(n, 1/2) \) and so,
\[ P(C_n) = P(\|N - n/2\| > \sqrt{n} \log n) \leq 2e^{-2n(\log n)^2/n} = 2e^{-2(\log n)^2}. \]

Thus,
\[ \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left( B_n(t) - \hat{B}_n \left( \frac{1}{2} \right) \right) \to 0, \] (14)

implying that \( V_n - X_n \to 0 \). Now,
\[ X_n - V_n \leq \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left( B_n^1(t) - \hat{B}_n^1 \left( \frac{N_1}{n} \right) \right) \vee \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left( B_n^2(t) - \hat{B}_n^2 \left( \frac{N_2}{n} \right) \right). \] (15)

To prove that
\[ \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left( \hat{B}_n(t) - \hat{B}_n \left( \frac{N}{n} \right) \right) \to 0, \] (16)

we use similar arguments, since
\[ P \left( \max_{t \in \left[ \frac{N}{2}, \frac{N}{n} \right]} \left| \hat{B}_n(t) - \hat{B}_n \left( \frac{N}{n} \right) \right| > \varepsilon \right) = P \left( \max_{k = N, \ldots, m} \frac{1}{\sqrt{2n}} \left| \sum_{j=N}^{k} \xi_j \right| > \varepsilon \right) \]
\[ \leq P \left( \max_{k = n/2 - \sqrt{\ln n}, \ldots, m \sqrt{\ln n}} \frac{1}{\sqrt{2n}} \left| \sum_{j=n/2 - \sqrt{\ln n}}^{k} \xi_j \right| > \varepsilon \right) + P(C_n) \to 0. \]

Hence, \( X_n - V_n \to 0 \), and so \( |X_n - V_n| \to 0 \). Together, the convergence results (14) and (16) imply that \( |U_n - V_n| \to 0 \).

Let us next prove that \( X_n \to 0 \). Again, we skip \( i \) from the notation. From (7),
\[ \frac{n/2 - N}{\sqrt{2n}} = \frac{n - TN}{2\sqrt{2n}} - \frac{1}{2} \hat{B}_n \left( \frac{N}{n} \right). \]
Let \( Y_i \), \( i = 1, 2 \), be a \([0, 1]\)-valued random element so that

\[
Y_i(t) := -\frac{1}{2} \hat{B}_i^i(t) + \hat{B}_n^n(t), \quad i = 1, 2.
\]

Since \( \hat{B}_n^n \rightarrow B^i \), it follows that \( Y_i \rightarrow B^i - 2^{-1} B^i(1/2) \), \( i = 1, 2 \). Let \( Y_n := (Y_n^1, Y_n^2) \). Then \( Y_n \) is a \([0, 1] \times [0, 1]\)-valued random element. Since \( Y_n^1 \) and \( Y_n^2 \) as well as \( B^1 \) and \( B^2 \) are independent, \( Y_n \Rightarrow (B^1, B^2) \). Appealing twice to the continuous mapping theorem shows that

\[
X_n = \max_{t \in [0, 1]} \left( Y_n^1(t) \wedge Y_n^2(t) \right) \Rightarrow \max_{t \in [0, 1]} \left[ \left( B^1(t) - \frac{1}{2} B^1 \left( \frac{1}{2} \right) \right) \wedge \left( B^2(t) - \frac{1}{2} B^2 \left( \frac{1}{2} \right) \right) \right].
\]

By (17), \( D_n \) converges in distribution to the same limit. \( \square \)

It would be interesting to find a more explicit representation for the law of the limiting distribution obtained in the above theorem, in other words for the law of

\[
\max_{t \in [0, 1]} \frac{1}{\sqrt{2}} \left[ B^1(t) - \frac{1}{2} B^1(1) - \left| B^2(t) - \frac{1}{2} B^2(1) \right| \right].
\]

Note also that the proof of the above theorem can be easily extended to an arbitrary, but fixed, number \( k \) of random sequences leading to the functional \( \max_{t \in [0, 1]} \min_{i = 1, \ldots, k} (B^i(t) - B^i(1)/2) \).

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**References**


