Probability Theory

Asymptotic behavior of the distribution of the stock price in models with stochastic volatility: the Hull–White model

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Received 18 June 2006; accepted 26 September 2006

Presented by Jean-Pierre Kahane

Abstract

In the present Note, we study the asymptotic behavior of the distribution density of the stock price process in the Hull–White model. The leading terms in the asymptotic expansions at zero and infinity are found for such a density and the corresponding error estimates are given. Similar problems are solved for time averages of the volatility process, which are also of interest in the study of Asian options. To cite this article: A. Gulisashvili, E.M. Stein, C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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Résumé


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Version française abrégée

Le modèle de Hull–White est l’un des modèles du prix de l’action à volatilité stochastique. La présente note étudie le modèle suivant qui est équivalent au modèle de Hull–White :

\[
\begin{align*}
\text{d}X_t &= \mu X_t \text{d}t + Y_t X_t \text{d}W_t, \\
\text{d}Y_t &= \nu Y_t \text{d}t + \xi Y_t \text{d}Z_t,
\end{align*}
\]

où \( \mu \in \mathbb{R} \), \( \nu \in \mathbb{R} \), \( \xi > 0 \), \( W_t \) et \( Z_t \) sont des browniens standards indépendants, \( Y_t \) le processus de volatilité et \( X_t \) le processus du prix de l’action. Les conditions initiales des processus \( X_t \) et \( Y_t \) sont respectivement notées \( x_0 \) et \( y_0 \).

On donne l’expression explicite de la partie principale du développement asymptotique de la densité de distribution.

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\(D_t(x; \mu, v, \xi, x_0, y_0)\) de la variable aléatoire \(X_t\). On étudie aussi le comportement asymptotique de la densité de distribution \(m_t(y; v, \xi, y_0)\) de la variable aléatoire

\[
\alpha_t = \alpha_t(v, \xi, y_0) = \left\{ \frac{1}{t} \int_0^t Y_s^2 \, ds \right\}^{1/2}.
\]

Par exemple si \(\mu = 0, \nu = 1/2, \xi = 1, x_0 = 1, y_0 = 1\), on a les expressions asymptotiques suivantes :

\[
m_t(y) = \frac{1}{\sqrt{\pi t y}} A_t(y) \left( 1 + O((\log y)^{-1/2}) \right), \quad y \to \infty,
\]

où \(A_t(y) = \exp(-\frac{u^2}{2t} + \frac{u y}{2t})\) et \(u_y\) vérifient \(\frac{\sinh u_y}{2u_y} = y^2\) :

\[
m_t(y) = \frac{\sqrt{2}}{\sqrt{\pi t}} \exp\left\{ \frac{\pi^2}{8t} \right\} y^{-1} \exp\left\{ -\frac{1}{2t y^2} \right\} (1 + O(y^2)), \quad y \to 0;
\]

\[
D_t(x) = \frac{1}{2\sqrt{\pi t}} x^{-2} (\log x)^{-1} A_t \left( \left( \frac{2}{t} \log x \right)^{1/2} \right) \left( 1 + O((\log x)^{-\delta}) \right), \quad x \to \infty
\]

pour tout \(\delta, 0 < \delta < 1/2\); et \(D_t(x) = x^{-3} D_t(\frac{1}{x})\). On obtient des formules similaires dans le cas général.

1. Introduction

Le Hull–White model est one of the standard stock price models with stochastic volatility (see [4]). Let \(X_t\) and \(Y_t\) be stochastic processes satisfying the following system of stochastic differential equations:

\[
\begin{aligned}
\frac{dX_t}{dt} &= \mu X_t \, dt + Y_t X_t \, dW_t, \\
\frac{dY_t}{dt} &= v Y_t \, dt + \xi Y_t \, dZ_t,
\end{aligned}
\]

where \(\mu \in \mathbb{R}, v \in \mathbb{R}, \xi > 0\), and \(W_t\) and \(Z_t\) are independent standard Brownian motions. In (1), the process \(Y_t\) is the volatility process and \(X_t\) is the stock price process. The initial conditions for the processes \(X_t\) and \(Y_t\) are denoted by \(x_0\) and \(y_0\), respectively. The model described in (1) is equivalent to the Hull–White model. We will denote by \(D_t(x) = D_t(x; \mu, v, \xi, x_0, y_0)\) the distribution density of the random variable \(X_t\), and by \(m_t(y) = m_t(y; v, \xi, y_0)\) the distribution density of the random variable

\[
\alpha_t(v, \xi, y_0) = \left\{ \frac{1}{t} \int_0^t Y_s^2 \, ds \right\}^{1/2}.
\]

The density \(m_t\) is called the mixing distribution density. The density \(D_t\) depends on \(\mu, v, \xi, x_0\), and \(y_0\), while the density \(m_t\) depends on \(v, \xi\), and \(y_0\). It is not hard to see that the following equality holds for \(m_t\) and \(D_t\):

\[
D(x; \mu, v, \xi, x_0, y_0) = \frac{1}{x_0 e^{\mu t}} \int_0^\infty L \left( t, y, \frac{x}{x_0 e^{\mu t}} \right) m_t(y; v, \xi, y_0) \, dy,
\]

where \(L\) is the lognormal density given by

\[
L(t, y, v) = \frac{1}{\sqrt{2\pi t y v}} \exp\left\{ -\frac{(\ln v + ty^2/2)^2}{2ty^2} \right\}.
\]

In the present Note we study the asymptotic behavior of the stock price distribution density \(D_t\) and the mixing distribution density \(m_t\). The mixing density \(m_t\) has numerous applications in the theory of Asian options (see [9]). We refer the reader to [3] for more information on stock price models with stochastic volatility and to [1,2,5,6,8,9] regarding the density \(m_t\). The asymptotics for the mixing density and the resulting asymptotics for the stock price distribution were first studied in [7] for the case of the model whose volatility process \(Y_t\) is mean-reverting Ornstein–Uhlenbeck process.
2. The asymptotics of the mixing distribution density

It will be assumed throughout the note that the coefficient $\mu$ is 0, since one can easily reduce matters to that case. We shall also set

$$\alpha = \frac{2\nu - \xi^2}{2\xi^2}. \quad (3)$$

For $y > 0$, we denote by $u_y$ the unique positive solution of the equation

$$\frac{\sinh(2u_y)}{2u_y} = y^2.$$ 

It is not hard to see that

$$u_y = \log y + \frac{1}{2} \log \log y + \log 2 + o(1)$$

as $y \to \infty$. Put

$$\Lambda_t(y) = \exp\left\{-\frac{u_y^2}{2t} + \frac{u_y}{2t}\right\},$$

where $t, y > 0$.

The asymptotic behavior of $m_t(y)$ as $y \to \infty$ is very roughly like $\exp\{-\frac{(\log y)^2}{2\xi^2}\}$. More precisely, the following theorem holds:

**Theorem 2.1.** If $-\infty < \nu < \infty$, $\xi > 0$, $y_0 > 0$, and $t > 0$, then

$$m_t(y) = c_t y^{\alpha-1} (\log y)^{\alpha/2} \Lambda_t\left(\frac{y}{y_0}\right) \left(1 + O((\log y)^{-1/2})\right), \quad y \to \infty,$$

where $\alpha$ is given by (3) and $c_t = \frac{1}{\sqrt{\pi t}} \exp\left\{-\frac{\alpha^2 \xi^2 t}{2}\right\}$.

We consider first the special case where $\nu = \frac{1}{2}$, $\xi = 1$, and $y_0 = 1$, and later the situation for more general $\nu$; then with $\xi = 1$, we have $\nu = \alpha + \frac{1}{2}$. We denote by $m_t^{(\alpha)}$ the mixing distribution density in this case and keep the notation $m_t$ when $\alpha = 0$ (i.e. $\nu = \frac{1}{2}$). Formula (4) then becomes

$$m_t(y) = \frac{1}{\sqrt{\pi t} y} \Lambda_t(y) \left(1 + O((\log y)^{-1/2})\right), \quad y \to \infty. \quad (5)$$

The proof of (5) and the general case proceeds as follows. First, one has in the special case $\alpha = 0$ an explicit integral formula,

$$m_t(y) = \frac{1}{\pi t} \exp\left\{\frac{\pi^2}{8t} y^2 \int_{-\infty}^{\infty} \exp\left\{-\frac{\cosh^2 u}{2ty^2}\right\} \cosh u \exp\left\{-\frac{u^2}{2t}\right\} \exp\left\{\frac{i\pi u}{2t}\right\} du\right\} \quad (6)$$

(see [1] where an equivalent formula is given). Alternatively, one can derive (6) by using the formula

$$u(x, t) = \int_{-\infty}^{\infty} e^{ix \sinh y} U(y, t) dy,$$

which transforms a solution $U$ of the heat equation to a solution of the equation

$$\frac{\partial u}{\partial t} = x^2 \frac{\partial^2 u}{\partial x^2} + x \frac{\partial u}{\partial x} - x^2 u. \quad (7)$$
Equations similar to that in (7) arise from the Feynman–Kac formula when computing the Laplace transform of \( m_t(y) \). With formula (6) in hand, one relies on the following lemma. For \( \varepsilon > 0 \), we define

\[
I(\varepsilon) = \int_{-\infty}^{\infty} e^{-\varepsilon(cosh u)^2} e^{-u^2/(2t)} \cosh u e^{i\pi u/(2t)} \, du.
\]

**Lemma 2.2.** The following equality holds:

\[
I(\varepsilon) = I_0(\varepsilon) \left( 1 + \left( \log \frac{1}{\varepsilon} \right) - \frac{1}{2} \right)
\]

as \( \varepsilon \to 0 \), where

\[
I_0(\varepsilon) = \left( \frac{\pi}{2} \right)^{1/2} \exp\left\{ -\frac{\pi^2}{8t} \right\} \exp\left\{ -\frac{N_\varepsilon^2}{2t} + \frac{N_\varepsilon}{2t} \right\} \varepsilon^{-1/2},
\]

and \( N_\varepsilon \) is the solution of \( \varepsilon \sinh (2N_\varepsilon) = N_\varepsilon t \).

The proof of the lemma requires that we deform the contour of integration for \( I_\varepsilon \) (the real one) into the complex \( u \)-plane, where the principal contributions are then given on the segments \([N_\varepsilon, N_\varepsilon + i\pi]\) and \([-N_\varepsilon, -N_\varepsilon + i\pi]\). With this lemma (and the corresponding asymptotics for \((\frac{d}{d\varepsilon})^k I(\varepsilon)\)) one obtains the desired result for \( \alpha = 0 \) and \( \alpha = 2k \), respectively, where \( k \) is a positive integer. We next use Dufresne’s recurrence formula (see [2]), which in our notation can be rewritten as follows:

\[
m_t^{(2r-\beta)}(\sqrt{\gamma}) = C_{r,\beta,t} y^{2r-\beta-1/2} \exp\left\{ -\frac{1}{2t y} \right\} \int_{y}^{\infty} (\tau - y)^{\beta - r - 1} \tau^{-\beta + 1/2} \exp\left\{ -\frac{1}{2t \tau} \right\} m_t^{(\beta)}(\sqrt{\tau}) \, d\tau,
\]

where

\[
C_{r,\beta,t} = \frac{(2t)^{r-\beta} \exp((-2r^2 + 2r \beta)t)}{\Gamma(\beta - r)}
\]

and \( r < \beta \). This allows us to deduce the asymptotics at \( \infty \) of \( m_t^{(\alpha)}(y) \) from those of \( m_t^{(2k)}(y) \), whenever \( \alpha < 2k \), and thus for all \( \alpha \). Finally, we drop the restriction \( \xi = 1, y_0 = 1 \), by observing that

\[
m_t(y; \nu, \xi, y_0) = \frac{1}{y_0} m_\xi:\nu,:y_0,1\left( \frac{y}{y_0} ; \frac{\nu}{\xi^2} , 1, 1 \right).
\]

Next, we formulate a theorem describing the asymptotics of the mixing distribution density as \( y \to 0 \). The analysis here is simpler than that for \( y \to \infty \). The result obtained is as follows.

**Theorem 2.3.** For each real \( \alpha \) and positive \( t \),

\[
m_t^{(\alpha)}(y) = b_{\alpha,t} y^{2\alpha-1} \exp\left\{ -\frac{1}{2t y^2} \right\} \left( 1 + O(y^2) \right), \quad y \to 0,
\]

where \( b_{\alpha,t} \) is an appropriate positive constant.

### 3. The asymptotics of the stock price distribution density

We can combine Theorem 2.1 with (2) to obtain the asymptotics of \( D_t(x) = D_t(x; \mu, \nu, \xi, x_0, y_0) \). In stating the result it suffices to consider the case where \( \mu = 0 \) and \( x_0 = 1 \), since (2) implies that

\[
D_t(x; \mu, \nu, \xi, x_0, y_0) = \frac{1}{x_0 e^{\mu t}} D_t(\frac{x}{x_0 e^{\mu t}}; 0, \nu, \xi, 1, y_0).
\]
Theorem 3.1. The following formula holds:

\[
D_t(x; 0, \nu, \xi, 1, y_0) = \frac{1}{2\sqrt{\pi t} y_0^{\alpha/2}} \exp\left\{ -\frac{\alpha^2 \xi^2 t}{2} \right\} x^{-2} (\log x)^{\alpha/2-1} (\log \log x)^{\alpha/2} \times \Lambda_t^{\delta^2} \left( \frac{1}{y_0} \left( \frac{2 \log x}{t} \right)^{1/2} \right) (1 + O((\log x)^{-\delta})), \quad x \to \infty,
\]

where \( \delta \) is any number with \( 0 < \delta < \frac{\alpha}{2} \) and \( \alpha \) is defined by (3). Moreover,

\[
D_t(x; 0, \nu, \xi, 1, y_0) = x^{-3} D_t(x; 0, \nu, \xi, 1, y_0).
\]

The proof requires a particular precise version of Laplace’s method. Suppose \( A(y) \) is a positive \( C^1 \) function on \([0, \infty)\) which satisfies \( |A'(y)| \leq cy^{\gamma}A(y) \), \( y > y_0 \), for some \( \gamma \) with \( 0 < \gamma \leq 1 \). Note that as a consequence

\[
\int_0^\infty A(y)e^{-by^2} \, dy < \infty \quad \text{for all } b > 0.
\]

Lemma 3.2. If \( c > 0 \), then

\[
\int_0^\infty A(y) \exp\left\{ -\left( \frac{x^2}{y^2} + c^2 y^2 \right) \right\} \, dy = \frac{\sqrt{\pi}}{2c} A(x^{1/2} c^{-1/2}) e^{-2cx} (1 + O(x^{-\delta})), \quad x \to \infty,
\]

for any \( \delta \) with \( 0 < \delta < \frac{\gamma}{2} \).

To apply the lemma we take \( A(y) \) to be the leading term of the asymptotic expansion of the function \( m_t(y) / y \).

Theorem 3.1 gives an asymptotic for \( D_t(x) \) very roughly of order \( x^{-2} \) as \( x \to \infty \), and roughly of order \( x^{-1} \) as \( x \to 0 \). This shows clearly that here the tails are “fatter” than in the case of the model where \( Y_t \) is mean-reverting Ornstein–Uhlenbeck process considered in [7]. There the corresponding sizes are of the order \( x^{-\gamma} \) for \( x \to \infty \), and \( x^{\gamma-3} \) when \( x \to 0 \), for an appropriate \( \gamma > 2 \).

References


