Available online at www.sciencedirect.com
ScienceDirect
C. R. Acad. Sci. Paris, Ser. I 343 (2006) 519–523
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Probability Theory

Asymptotic behavior of the distribution of the stock price in models with stochastic volatility: the Hull–White model

Archil Gulisashvili [a], Elias M. Stein [b]

[a] *Department of Mathematics, Ohio University, Athens, OH 45701, USA*
[b] *Department of Mathematics, Princeton University, Princeton, NJ 08540, USA*

Received 18 June 2006; accepted 26 September 2006

Presented by Jean-Pierre Kahane

Abstract

In the present Note, we study the asymptotic behavior of the distribution density of the stock price process in the Hull–White model. The leading terms in the asymptotic expansions at zero and infinity are found for such a density and the corresponding error estimates are given. Similar problems are solved for time averages of the volatility process, which are also of interest in the study of Asian options. ***To cite this article: A. Gulisashvili, E.M. Stein, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Comportement asymptotique de la distribution du prix de l'action dans les modéles à volatilité stochastique : le modèle de Hull–White. La présente Note étudie le comportement asymptotique de la densité de distribution du processus du prix de l'action dans le modèle de Hull–White. On determine la partie principale dans le développement asymptotique en zéro et en l'infini pour une telle densité et on estime l'erreur correspondante. Des problèmes similaires se résolvent pour les moyennes temporelles du processus de volatilité qui sont aussi intéressants dans l'étude des options asiatiques. ***Pour citer cet article : A. Gulisashvili, E.M. Stein, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

Le modèle de Hull–White est l'un des modèles du prix de l'action à volatilité stochastique. La présente note étudie le modéle suivant qui est équivalent au modèle de Hull–White :

$$\begin{cases} \mathrm{d}X_t = \mu X_t\,\mathrm{d}t + Y_t X_t\,\mathrm{d}W_t, \\ \mathrm{d}Y_t = \nu Y_t\,\mathrm{d}t + \xi Y_t\,\mathrm{d}Z_t, \end{cases}$$

où $\mu \in R$, $\nu \in R$, $\xi > 0$, W_t et Z_t sont des browniens standards indépendants, Y_t le processus de volatilité et X_t le processus du prix de l'action. Les conditions initiales des processus X_t et Y_t sont respectivement notées x_0 et y_0. On donne l'expression explicite de la partie principale du développement asymptotique de la densité de distribution

E-mail addresses: guli@math.ohiou.edu (A. Gulisashvili), stein@math.princeton.edu (E.M. Stein).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.09.029

$D_t(x;\mu,\nu,\xi,x_0,y_0)$ de la variable aléatoire X_t. On étudie aussi le comportement asymptotique de la densité de distribution $m_t(y;\nu,\xi,y_0)$ de la variable aléatoire

$$\alpha_t=\alpha_t(\nu,\xi,y_0)=\left\{\frac{1}{t}\int_0^t Y_s^2\,\mathrm{d}s\right\}^{1/2}.$$

Par exemple si $\mu=0$, $\nu=\frac{1}{2}$, $\xi=1$, $x_0=1$, $y_0=1$, on a les expressions asymptotiques suivantes :

$$m_t(y)=\frac{1}{\sqrt{\pi t}\,y}\Lambda_t(y)\big(1+\mathrm{O}\big((\log y)^{-1/2}\big)\big),\quad y\to\infty,$$

où $\Lambda_t(y)=\exp\{-\frac{u_y^2}{2t}+\frac{u_y}{2t}\}$ et u_y vérifient $\frac{\sinh u_y}{2u_y}=y^2$;

$$m_t(y)=\frac{\sqrt{2}}{\sqrt{\pi t}}\exp\left\{\frac{\pi^2}{8t}\right\}y^{-1}\exp\left\{-\frac{1}{2ty^2}\right\}\big(1+\mathrm{O}(y^2)\big),\quad y\to 0;$$

$$D_t(x)=\frac{1}{2\sqrt{\pi t}}x^{-2}(\log x)^{-1}\Lambda_t\left(\left(\frac{2}{t}\log x\right)^{1/2}\right)\big(1+\mathrm{O}\big((\log x)^{-\delta}\big)\big),\quad x\to\infty$$

pour tout δ, $0<\delta<\frac{1}{2}$; et $D_t(x)=x^{-3}D_t(\frac{1}{x})$. On obtient des formules similaires dans le cas général.

1. Introduction

The Hull–White model is one of the standard stock price models with stochastic volatility (see [4]). Let X_t and Y_t be stochastic processes satisfying the following system of stochastic differential equations:

$$\begin{cases}\mathrm{d}X_t=\mu X_t\,\mathrm{d}t+Y_tX_t\,\mathrm{d}W_t,\\ \mathrm{d}Y_t=\nu Y_t\,\mathrm{d}t+\xi Y_t\,\mathrm{d}Z_t,\end{cases}\tag{1}$$

where $\mu\in R$, $\nu\in R$, $\xi>0$, and W_t and Z_t are independent standard Brownian motions. In (1), the process Y_t is the volatility process and X_t is the stock price process. The initial conditions for the processes X_t and Y_t are denoted by x_0 and y_0, respectively. The model described in (1) is equivalent to the Hull–White model. We will denote by $D_t(x)=D_t(x;\mu,\nu,\xi,x_0,y_0)$ the distribution density of the random variable X_t, and by $m_t(y)=m_t(y;\nu,\xi,y_0)$ the distribution density of the random variable

$$\alpha_t(\nu,\xi,y_0)=\left\{\frac{1}{t}\int_0^t Y_s^2\,\mathrm{d}s\right\}^{1/2}.$$

The density m_t is called the mixing distribution density. The density D_t depends on μ, ν, ξ, x_0, and y_0, while the density m_t depends on ν, ξ, and y_0. It is not hard to see that the following equality holds for m_t and D_t:

$$D(x;\mu,\nu,\xi,x_0,y_0)=\frac{1}{x_0\mathrm{e}^{\mu t}}\int_0^\infty L\left(t,y,\frac{x}{x_0\mathrm{e}^{\mu t}}\right)m_t(y;\nu,\xi,y_0)\,\mathrm{d}y,\tag{2}$$

where L is the lognormal density given by

$$L(t,y,v)=\frac{1}{\sqrt{2\pi t}\,yv}\exp\left\{-\frac{(\ln v+ty^2/2)^2}{2ty^2}\right\}.$$

In the present Note we study the asymptotic behavior of the stock price distribution density D_t and the mixing distribution density m_t. The mixing density m_t has numerous applications in the theory of Asian options (see [9]). We refer the reader to [3] for more information on stock price models with stochastic volatility and to [1,2,5,6,8,9] regarding the density m_t. The asymptotics for the mixing density and the resulting asymptotics for the stock price distribution were first studied in [7] for the case of the model whose volatility process Y_t is mean-reverting Ornstein–Uhlenbeck process.

2. The asymptotics of the mixing distribution density

It will be assumed throughout the note that the coefficient μ is 0, since one can easily reduce matters to that case. We shall also set

$$\alpha = \frac{2\nu - \xi^2}{2\xi^2}. \tag{3}$$

For $y > 0$, we denote by u_y the unique positive solution of the equation

$$\frac{\sinh(2u_y)}{2u_y} = y^2.$$

It is not hard to see that

$$u_y = \log y + \frac{1}{2}\log\log y + \log 2 + \mathrm{o}(1)$$

as $y \to \infty$. Put

$$\Lambda_t(y) = \exp\left\{-\frac{u_y^2}{2t} + \frac{u_y}{2t}\right\},$$

where $t, y > 0$.

The asymptotic behavior of $m_t(y)$ as $y \to \infty$ is very roughly like $\exp\{-\frac{(\log y)^2}{2t\xi^2}\}$. More precisely, the following theorem holds:

Theorem 2.1. *If $-\infty < \nu < \infty$, $\xi > 0$, $y_0 > 0$, and $t > 0$, then*

$$m_t(y) = c_t y^{\alpha-1}(\log y)^{\alpha/2}\Lambda_{t\xi^2}\left(\frac{y}{y_0}\right)\left(1 + \mathrm{O}\left((\log y)^{-1/2}\right)\right), \quad y \to \infty, \tag{4}$$

where α is given by (3) *and*

$$c_t = \frac{1}{y_0^\alpha \xi\sqrt{\pi t}}\exp\left\{-\frac{\alpha^2\xi^2 t}{2}\right\}.$$

We consider first the special case where $\nu = \frac{1}{2}$, $\xi = 1$, and $y_0 = 1$, and later the situation for more general ν; then with $\xi = 1$, we have $\nu = \alpha + \frac{1}{2}$. We denote by $m_t^{(\alpha)}$ the mixing distribution density in this case and keep the notation m_t when $\alpha = 0$ (i.e. $\nu = \frac{1}{2}$). Formula (4) then becomes

$$m_t(y) = \frac{1}{\sqrt{\pi t}\, y}\Lambda_t(y)\left(1 + \mathrm{O}\left((\log y)^{-1/2}\right)\right), \quad y \to \infty. \tag{5}$$

The proof of (5) and the general case proceeds as follows. First, one has in the special case $\alpha = 0$ an explicit integral formula,

$$m_t(y) = \frac{1}{\pi t}\exp\left\{\frac{\pi^2}{8t}\right\}y^{-2}\int_{-\infty}^{\infty}\exp\left\{-\frac{\cosh^2 u}{2ty^2}\right\}\cosh u \exp\left\{-\frac{u^2}{2t}\right\}\exp\left\{\frac{\mathrm{i}\pi u}{2t}\right\}\mathrm{d}u \tag{6}$$

(see [1] where an equivalent formula is given).

Alternatively, one can derive (6) by using the formula

$$u(x,t) = \int_{-\infty}^{\infty} \mathrm{e}^{\mathrm{i}x\sinh y}U(y,t)\,\mathrm{d}y,$$

which transforms a solution U of the heat equation to a solution of the equation

$$\frac{\partial u}{\partial t} = x^2\frac{\partial^2 u}{\partial x^2} + x\frac{\partial u}{\partial x} - x^2 u. \tag{7}$$

Equations similar to that in (7) arise from the Feynman–Kac formula when computing the Laplace transform of $m_t(y)$. With formula (6) in hand, one relies on the following lemma. For $\varepsilon > 0$, we define

$$I(\varepsilon) = \int_{-\infty}^{\infty} \mathrm{e}^{-\varepsilon(\cosh u)^2} \mathrm{e}^{-u^2/(2t)} \cosh u \mathrm{e}^{i\pi u/(2t)} \, \mathrm{d}u.$$

Lemma 2.2. *The following equality holds*:

$$I(\varepsilon) = I_0(\varepsilon)\left(1 + \left(\log \frac{1}{\varepsilon}\right)^{-1/2}\right)$$

as $\varepsilon \to 0$, *where*

$$I_0(\varepsilon) = \left(\frac{\pi}{2}\right)^{1/2} \exp\left\{-\frac{\pi^2}{8t}\right\} \exp\left\{-\frac{N_\varepsilon^2}{2t} + \frac{N_\varepsilon}{2t}\right\} \varepsilon^{-1/2},$$

and N_ε *is the solution of* $\varepsilon \sinh(2N_\varepsilon) = \frac{N_\varepsilon}{t}$.

The proof of the lemma requires that we deform the contour of integration for I_ε (the real one) into the complex u-plane, where the principal contributions are then given on the segments $[N_\varepsilon, N_\varepsilon + \mathrm{i}\pi]$ and $[-N_\varepsilon, -N_\varepsilon + \mathrm{i}\pi]$. With this lemma (and the corresponding asymptotics for $(\frac{\mathrm{d}}{\mathrm{d}\varepsilon})^k I(\varepsilon)$) one obtains the desired result for $\alpha = 0$ and $\alpha = 2k$, respectively, where k is a positive integer. We next use Dufresne's recurrence formula (see [2]), which in our notation can be rewritten as follows:

$$m_t^{(2r-\beta)}\left(\sqrt{y}\right) = C_{r,\beta,t}\, y^{2r-\beta-1/2} \exp\left\{-\frac{1}{2ty}\right\} \int_y^{\infty} (\tau - y)^{\beta-r-1} \tau^{-\beta+1/2} \exp\left\{\frac{1}{2t\tau}\right\} m_t^{(\beta)}\left(\sqrt{\tau}\right) \mathrm{d}\tau,$$

where

$$C_{r,\beta,t} = \frac{(2t)^{r-\beta} \exp\{(-2r^2 + 2r\beta)t\}}{\Gamma(\beta - r)}$$

and $r < \beta$. This allows us to deduce the asymptotics at ∞ of $m_t^{(\alpha)}(y)$ from those of $m_t^{(2k)}(y)$, whenever $\alpha < 2k$, and thus for all α. Finally, we drop the restriction $\xi = 1$, $y_0 = 1$, by observing that

$$m_t(y; \nu, \xi, y_0) = \frac{1}{y_0} m_{t\xi^2}\left(\frac{y}{y_0}; \frac{\nu}{\xi^2}, 1, 1\right).$$

Next, we formulate a theorem describing the asymptotics of the mixing distribution density as $y \to 0$. The analysis here is simpler than that for $y \to \infty$. The result obtained is as follows.

Theorem 2.3. *For each real* α *and positive* t,

$$m_t^{(\alpha)}(y) = b_{\alpha,t}\, y^{2\alpha-1} \exp\left\{-\frac{1}{2ty^2}\right\}\left(1 + \mathrm{O}\left(y^2\right)\right), \quad y \to 0,$$

where $b_{\alpha,t}$ *is an appropriate positive constant.*

3. The asymptotics of the stock price distribution density

We can combine Theorem 2.1 with (2) to obtain the asymptotics of $D_t(x) = D_t(x; \mu, \nu, \xi, x_0, y_0)$. In stating the result it suffices to consider the case where $\mu = 0$ and $x_0 = 1$, since (2) implies that

$$D_t(x; \mu, \nu, \xi, x_0, y_0) = \frac{1}{x_0 \mathrm{e}^{\mu t}} D_t\left(\frac{x}{x_0 \mathrm{e}^{\mu t}}; 0, \nu, \xi, 1, y_0\right).$$

Theorem 3.1. *The following formula holds*:

$$D_t(x;0,\nu,\xi,1,y_0)=\frac{1}{2\sqrt{\pi t}\,y_0^{\alpha}\xi t^{\alpha/2}}\exp\left\{-\frac{\alpha^2\xi^2 t}{2}\right\}x^{-2}(\log x)^{\alpha/2-1}(\log\log x)^{\alpha/2}$$
$$\times\Lambda_{t\xi^2}\left(\frac{1}{y_0}\left(\frac{2\log x}{t}\right)^{1/2}\right)\bigl(1+\mathrm{O}\bigl((\log x)^{-\delta}\bigr)\bigr),\quad x\to\infty,$$

where δ is any number with $0<\delta<\frac{1}{2}$ and α is defined by (3). *Moreover,*

$$D_t(x;0,\nu,\xi,1,y_0)=x^{-3}D_t\left(\frac{1}{x};0,\nu,\xi,1,y_0\right).$$

The proof requires a particular precise version of Laplace's method. Suppose $A(y)$ is a positive C^1 function on $[0,\infty)$ which satisfies $|A'(y)|\leqslant cy^{-\gamma}A(y)$, $y>y_0$, for some γ with $0<\gamma\leqslant 1$. Note that as a consequence $\int_0^\infty A(y)\mathrm{e}^{-by^2}\,\mathrm{d}y<\infty$ for all $b>0$.

Lemma 3.2. *If $c>0$, then*

$$\int_0^\infty A(y)\exp\left\{-\left(\frac{x^2}{y^2}+c^2y^2\right)\right\}\mathrm{d}y=\frac{\sqrt{\pi}}{2c}A\bigl(x^{1/2}c^{-1/2}\bigr)\mathrm{e}^{-2cx}\bigl(1+\mathrm{O}\bigl(x^{-\delta}\bigr)\bigr),\quad x\to\infty,$$

for any δ with $0<\delta<\frac{\gamma}{2}$.

To apply the lemma we take $A(y)$ to be the leading term of the asymptotic expansion of the function $\frac{m_t(y)}{y}$.

Theorem 3.1 gives an asymptotic for $D_t(x)$ very roughly of order x^{-2} as $x\to\infty$, and roughly of order x^{-1} as $x\to 0$. This shows clearly that here the tails are "fatter" than in the case of the model where Y_t is mean-reverting Ornstein–Uhlenbeck process considered in [7]. There the corresponding sizes are of the order $x^{-\gamma}$ for $x\to\infty$, and $x^{\gamma-3}$ when $x\to 0$, for an appropriate $\gamma>2$.

References

[1] L. Alili, J.C. Gruet, An explanation of a generalized Boujerol's identity in terms of hyperbolic geometry, in: M. Yor (Ed.), Exponential Functionals and Principal Values Related to Brownian Motion, in: Biblioteca de la Revista Matemática Ibero-Americana, Revista Matemática Iberoamericana, Madrid, 1997, pp. 15–33.
[2] D. Dufresne, The integral of geometric Brownian motion, Adv. Appl. Probab. 33 (2001) 223–241.
[3] J.-P. Fouque, G. Papanicolaou, K.R. Sircar, Derivatives in Financial Markets with Stochastic Volatility, Cambridge University Press, Cambridge, 2000.
[4] J. Hull, A. White, The pricing of options on assets with stochastic volatilities, J. Finance 42 (1987) 281–300.
[5] H. Matsumoto, M. Yor, On Dufresne's relation between the probability laws of exponential functionals of Brownian motions with different drifts, Adv. Appl. Probab. 35 (2003) 184–206.
[6] M. Schröder, On the integral of geometric Brownian motion, Adv. Appl. Probab. 35 (2003) 159–183.
[7] E.M. Stein, J. Stein, Stock price distributions with stochastic volatility: An analytic approach, Rev. Financial Stud. 4 (1991) 727–752.
[8] M. Yor, On some exponential functionals of Brownian motion, Adv. Appl. Probab. 24 (1992) 509–531.
[9] M. Yor, Exponential Functionals of Brownian Motion and Related Processes, Springer-Verlag, Berlin, 2001.