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Partial Differential Equations

Remarks on global approximate controllability for the 2-D Navier–Stokes system with Dirichlet boundary conditions

Sergio Guerrero [a], Oleg Yurievich Imanuvilov [b], Jean-Pierre Puel [c]

[a] *Laboratoire Jacques-Louis Lions, université Pierre et Marie Curie, boîte corrier, 187, 75035 Paris cedex 05, France*
[b] *Department of Mathematics, Colorado State University, 101 Weber Building, Fort Collins, CO 80523-1874, USA*
[c] *Laboratoire de mathématiques de Versailles, université de Versailles-St-Quentin, 45, avenue des États Unis, 78035 Versailles, France*

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Abstract

This Note deals with the two-dimensional Navier–Stokes system. In this context, we prove a result concerning its global approximate controllability by means of boundary controls. ***To cite this article: S. Guerrero et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Remarques sur la contrôlabilité globale approchée du système 2-D de Navier–Stokes avec conditions au bord de type Dirichlet. Cette Note concerne le système de Navier–Stokes en dimension 2. Nous montrons un résultat de contrôlabilité approchée globale à l'aide de contrôles frontière. ***Pour citer cet article : S. Guerrero et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

On considère le système de Navier–Stokes

$$\begin{cases} \partial_t u - \Delta u + (u,\nabla)u = \nabla p + f, \quad \nabla\cdot u = 0, & (t,x)\in Q := (0,T)\times\Omega,\\ u(t,0,x_2) = 0, & (t,x_2)\in(0,T)\times(0,1),\\ u(0,x) = u_0, & x=(x_1,x_2)\in\Omega, \end{cases} \tag{1}$$

posé dans un carré $\Omega = (0,1)\times(0,1)$ de $\mathbb{R}^2$ où une condition de Dirichlet homogène est donnée sur le seul côté $\{0\}\times(0,1)$, la trace de la vitesse sur les autres côtés du carré n'étant pas spécifiée et correspondant au contrôle frontière. Le but du présent travail est de montrer un résultat de contrôlabilité approchée globale pour ce système. De manière précise, nous montrons le résultat suivant.

E-mail addresses: guerrero@ann.jussieu.fr (S. Guerrero), oleg@math.colostate.edu (O.Y. Imanuvilov), jppuel@cmapx.polytechnique.fr (J.-P. Puel).

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Théorème 0.1. *Soit $u_0 \in H^1(\Omega)^2$ satisfaisant* (4) *et soit $f \in L^2(Q)^2$. Alors il existe une suite $\{f_\epsilon\}_{\epsilon>0} \subset L^2(Q)^2$ telle que*

$$f_\epsilon \to f \quad \text{in } L^{p_0}(0,T;V'),\ p_0 \in (1, 8/7),$$

et il existe une solution $u_\epsilon \in L^2(0,T;H^2(\Omega)^2) \cap H^1(0,T;L^2(\Omega)^2) := H^{1,2}(Q)$ du problème de contrôlabilité

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon + (u_\epsilon, \nabla)u_\epsilon + \nabla p_\epsilon = f_\epsilon, \quad \nabla \cdot u_\epsilon = 0, & (t,x) \in Q, \\ u_\epsilon(t,0,x_2) = 0, & (t,x_2) \in (0,T) \times (0,1), \\ u_\epsilon(0,x) = u_0, \quad u_\epsilon(T,x) = 0, & x \in \Omega. \end{cases} \tag{2}$$

La preuve s'obtient par une construction de la fonction u_ϵ en plusieurs étapes et s'inspire de la méthode du retour de Coron [1,2]. Tout d'abord sur un long intervalle de temps, on n'exerce aucun contrôle c'est-à-dire qu'on prend la solution avec condition de Dirichlet homogène sur tout le bord. Puis sur un intervalle de temps très court on approche la vitesse par une vitesse à support compact dans Ω en perturbant le second membre f. Ensuite, toujours sur un court intervalle de temps, repartant de cette nouvelle vitesse (initiale), on construit une solution qui atteint exactement une solution particulière N^2U avec N grand et U de la forme $U(t,x) = (0, z(t,x_1))$ où z est solution d'une équation de type chaleur contrôlée sur le bord. Enfin, on termine en utilisant les résultats désormais connus sur la contrôlabilité à zéro de l'équation de la chaleur.

1. Introduction and main results

In this Note, we deal with the following 2-D Navier–Stokes system:

$$\begin{cases} \partial_t u - \Delta u + (u, \nabla)u = \nabla p + f, \quad \nabla \cdot u = 0, & (t,x) \in Q := (0,T) \times \Omega, \\ u(t,0,x_2) = 0, & (t,x_2) \in (0,T) \times (0,1), \\ u(0,x) = u_0, & x = (x_1,x_2) \in \Omega. \end{cases} \tag{3}$$

Here Ω is the open set given by $\Omega = \{x = (x_1,x_2)\colon\ x_1 \in (0,1),\ x_2 \in (0,1)\}$, $f \in L^2(Q)^2$ is a given function and $u_0 \in H^1(\Omega)^2$ satisfies

$$\nabla \cdot u_0 = 0, \quad x \in \Omega \quad \text{and} \quad u_0(0,x_2) = 0, \quad x_2 \in (0,1). \tag{4}$$

Observe that in system (3) we did not provide the values of u on $(\{1\} \times (0,1)) \cup ((0,1) \times \{0,1\})$. In fact, this is the part of $\partial\Omega$ where the control is acting.

We introduce some spaces which are usual in the context of incompressible fluids:

$$V = \{w \in H^1(\Omega)^2\colon\ \nabla \cdot w = 0 \text{ in } \Omega,\ w = 0 \text{ on } \{0,1\} \times (0,1)\}$$

and

$$H = \{w \in L^2(\Omega)^2\colon\ \nabla \cdot w = 0 \text{ in } \Omega,\ w \cdot n = 0 \text{ on } \{0,1\} \times (0,1)\}.$$

Our main goal in this Note is to prove a result concerning the global approximate controllability for system (3). A result of local exact controllability has been given in [4]. We present this result in the following theorem:

Theorem 1.1. *Let $u_0 \in H^1(\Omega)^2$ satisfy* (4) *and $f \in L^2(Q)^2$. Then, there exists a sequence $\{f_\epsilon\}_{\epsilon>0} \subset L^2(Q)^2$ such that*

$$f_\epsilon \to f \quad \text{in } L^{p_0}(0,T;V'),\ p_0 \in (1, 8/7),$$

and there exists at least one solution $u_\epsilon \in L^2(0,T;H^2(\Omega)^2) \cap H^1(0,T;L^2(\Omega)^2) := H^{1,2}(Q)$ to the controllability problem

$$\begin{cases} \partial_t u_\epsilon - \Delta u_\epsilon + (u_\epsilon, \nabla)u_\epsilon + \nabla p_\epsilon = f_\epsilon, \quad \nabla \cdot u_\epsilon = 0, & (t,x) \in Q, \\ u_\epsilon(t,0,x_2) = 0, & (t,x_2) \in (0,T) \times (0,1), \\ u_\epsilon(0,x) = u_0, \quad u_\epsilon(T,x) = 0, & x \in \Omega. \end{cases} \tag{5}$$

Let L be a large positive constant. We extend the function u_0 on the domain $(0,1)\times(-L/2,L/2)$ in such a way that u_0 is a divergence free function, and for small $\tilde{\delta}>0$ we have

$$u_0(x)=0 \quad \text{for } x_2\in(-L/2,-\tilde{\delta})\cup(1+\tilde{\delta},L/2). \tag{6}$$

After that, we extend u_0 on $\mathcal{G}=(0,1)\times\mathbf{R}$ as a L periodic function.

The way we find the function u_ϵ is based on Coron's return method (see [1,2]) and it is constructive. The general idea is first to drive our solution to a special trajectory of the Navier–Stokes system which we call N^2U, where N is a large constant. Then, due to the particular structure of U, we can reduce the problem of driving N^2U to zero to a boundary controllability problem for a linear heat equation (which is a well-known result).

In the next section, we will construct the function U as well as two other functions $\tilde{y}$ and W (solutions of some parabolic and hyperbolic partial differential equations) which will be useful for driving our solution of system (5) to N^2U. Finally, in the last section, we will sketch the proof of Theorem 1.1.

2. Construction of the functions U, $\tilde{y}$ and W

Let $z=z(t,x_1)$ solve the following problem associated to a linear heat equation:

$$\begin{cases}\partial_t z-\partial^2_{x_1x_1}z=c(t), & (t,x_1)\in(0,T)\times(0,1),\\ z(t,0)=0,\quad z(t,1)=w(t), & t\in(0,T),\\ z(0,x_1)=0, & x_1\in(0,1).\end{cases} \tag{7}$$

Here $c(t)$ is a constant for each t with $c\in C^2[0,T]$, $c(0)\neq 0$ and w satisfies

$$w\in C^\infty[0,T],\quad w(0)=0,\quad w'(0)=c(0),\quad w''(0)=c'(0).$$

Under these conditions, one can prove that z satisfies $z\in C^2([0,T]\times[\gamma,1])\ \forall\gamma>0$.

Using this function, we construct $U(t,x)=(0,z(t,x_1))$ and $q(t,x)=x_2c(t)$ for $(t,x)\in(0,T)\times\mathcal{G}$, which solves

$$\begin{cases}\partial_t U-\Delta U+(U,\nabla)U+\nabla q=0,\quad \nabla\cdot U=0, & (t,x)\in(0,T)\times\mathcal{G},\\ U(t,0,x_2)=0, & (t,x_2)\in(0,T)\times\mathbf{R},\\ U(t,x_1,x_2)=U(t,x_1,x_2+L), & (t,x_1,x_2)\in(0,T)\times\mathcal{G},\\ U(0,x)=0, & x\in\mathcal{G}.\end{cases} \tag{8}$$

Next, we find a function $\tilde{y}$ which solves the following null controllability problem:

$$\begin{cases}\partial_t\tilde{y}+N^2(U,\nabla)\tilde{y}+N^2(\tilde{y},\nabla)U=0, & (t,x)\in(0,T)\times\mathcal{G},\\ \tilde{y}(t,0,x_2)=0, & (t,x_2)\in(0,T)\times\mathbf{R},\\ \tilde{y}(0,x)=\tilde{u}_0\quad x\in\mathcal{G},\quad \tilde{y}(1/N,x)=0, & x\in\Omega.\end{cases} \tag{9}$$

More precisely, we have:

Proposition 2.1. *Let us suppose that there exists $\gamma>0$ such that* $\operatorname{supp}\tilde{u}_0\subset\{x\in\Omega:\ x_1\geqslant\gamma\}$. *Then there exists a solution $\tilde{y}$ to the problem* (9) *and a positive constant C, such that*

$$\|\tilde{y}\|_{C^2(\overline{Q}_{1/N})^2}+\frac{1}{N}\|\partial_t\tilde{y}\|_{C^2(\overline{Q}_{1/N})^2}\leqslant C\|\tilde{u}_0\|_{C^3(\overline{\mathcal{G}})^2}. \tag{10}$$

Here, we have denoted

$$Q_{1/N}=(0,1/N)\times\mathcal{G}=(0,1/N)\times(0,1)\times\mathbf{R}.$$

In order to prove this proposition, we make a particular choice for the function z in (7) and we take $\tilde{y}$ depending in a very particular way of the primitive in time of z. All the details of the proof will be provided in [7].

Once $\tilde{y}$ is known, we define (W,r) as the solution to the following Stokes problem:

$$\begin{cases}\partial_t W-\Delta W+\nabla r=0,\quad \nabla\cdot W=\nabla\cdot\tilde{y}, & (t,x)\in(0,T)\times\mathcal{G},\\ W(t,0,x_2)=W(t,1,x_2)=0, & (t,x_2)\in(0,T)\times\mathbf{R},\\ W(t,x_1,x_2)=W(t,x_1,x_2+L), & (t,x_1,x_2)\in(0,T)\times\mathcal{G},\\ W(0,x)=0, & x\in\mathcal{G}.\end{cases} \tag{11}$$

For this function we can prove some estimates:

Proposition 2.2. *The following a priori estimates hold true*:

- *For any* $p \in (1, \infty)$
$$\|W\|_{L^p(Q_{2/N})^2} \leqslant \widetilde{C}(p, N)\|\tilde{u}_0\|_{C^1(\overline{\Omega})^2} \quad \text{with } \widetilde{C}(p, N) \to 0 \text{ as } N \to +\infty. \tag{12}$$
- *There exists a positive constant* $C > 0$ *independent of* N *such that*
$$\|W\|_{C([0,2/N];L^2(\mathcal{G})^2)} + \|\partial_{x_2} W\|_{C([0,2/N];L^2(\mathcal{G})^2)} \leqslant \frac{C}{N^{1/8}}\|\tilde{u}_0\|_{C^3(\overline{\Omega})^2}. \tag{13}$$

In order to establish estimates (12) and (13), the essential tool is a regularity result for the pressure as long as we consider *energy solutions* of the Stokes system together with the regularity results of [6]. More precisely, let us consider $w \in L^2(0, T; H^1(\Omega)^2) \cap C([0, T]; L^2(\Omega)^2)$ (together with some pressure h) the weak solution of the following Stokes system:

$$\begin{cases} w_t - \Delta w + \nabla h = f, \quad \nabla \cdot w = 0, & (t, x) \in (0, T) \times \mathcal{G}, \\ w(t, 0, x_2) = w(t, 1, x_2) = 0, & (t, x_2) \in (0, T) \times \mathbf{R}, \\ w(t, x_1, x_2) = w(t, x_1, x_2 + L), & (t, x_1, x_2) \in (0, T) \times \mathcal{G}, \\ w(0, x) = w_0, & x \in \mathcal{G}. \end{cases} \tag{14}$$

Here, $T > 0$ is a positive number and $\mathcal{G}$ was defined in the introduction. The regularity result for the pressure is presented in the following lemma:

Lemma 2.1. *Let* $w_0 \in H$, $f \in L^2(0, T; H^{-1}(\mathcal{G})^2)$. *Then, the pressure term* h *in* (14) *satisfies* $h \in H^{-1/4}(0, T; L^2(\mathcal{G}))$. *Moreover, there exists a positive constant* $C = C(\mathcal{G})$ *independent of* T *such that*
$$\|h\|_{H^{-1/4}(0,T;L^2(\mathcal{G})^2)} \leqslant C\big(\|w_0\|_{L^2(\mathcal{G})} + \|f\|_{L^2(0,T;H^{-1}(\mathcal{G})^2)}\big). \tag{15}$$

This lemma extends a local (in time) result which was proved in [3].

All the details of the proof of this lemma and that of Proposition 2.2 are provided in [7].

3. Proof of Theorem 1.1

For $\epsilon > 0$ let $\delta_0(\epsilon) > 0$ be sufficiently small so that for every $\delta \in (0, \delta_0(\epsilon))$ we have
$$\|f\|_{L^{p_0}(T-3\delta,T;V')} \leqslant \epsilon/10.$$

Let $u \in H^{1,2}(Q)$ be the solution to the Navier–Stokes system with homogeneous Dirichlet boundary conditions, initial condition u_0 and right-hand side f. We divide the proof in several steps, depending on the time interval we are working on:

- First, between $t = 0$ and $t = T - 3\delta$, we do not exert any control. So, in this interval, $u_\epsilon \equiv u$.
- Next, in the interval $[T - 3\delta, T - 2\delta]$, we consider a function $u_{0,\epsilon_1} \in V \cap C_0^\infty(\Omega)$ such that
$$\big\|u_{0,\epsilon_1} - u(T - 3\delta)\big\|_{H^1(\Omega)^2} \leqslant \delta^3.$$

Then, the expression of u_ϵ is precisely given by:
$$u_\epsilon(t, x) = \frac{t - T + 3\delta}{\delta} u_{0,\epsilon_1}(x) - \frac{t - T + 2\delta}{\delta} u(T - 3\delta, x), \quad (t, x) \in [T - 3\delta, T - 2\delta] \times \Omega.$$

- In the next step, on the segment $[T - 2\delta, T - 2\delta + 2/N]$, we look for the solution u_ϵ in the form
$$u_\epsilon(t, x) = N^2\widetilde{U}(t, x) + \mathbf{y}(t, x) - \widetilde{W}(t, x), \qquad p_\epsilon(t, x) = \tilde{r}(t, x)$$

for $(t, x) \in [T - 2\delta, T - 2\delta + 2/N] \times \Omega$. Here,
$$\widetilde{U}(t, x) = U(t - T + 2\delta, x), \qquad \mathbf{y}(t, x) = \tilde{y}(t - T + 2\delta, x),$$

where U is the solution to problem (8), $\tilde{y}$ is solution to the problem (9) with initial condition $\tilde{u}_0 = u_{0,\epsilon_1}$ (which obviously satisfies the hypothesis of Proposition 2.1) and

$$\widetilde{W}(t,x) = \theta(t-T+2\delta)W(t-T+2\delta,x), \qquad \tilde{r}(t,x) = \theta(t)r(t-T+2\delta,x),$$

where (W,r) is solution of (11) and $\theta = \theta(t) \in C^2([0,2/N])$ satisfies

$$\theta(t) = 1, \quad t \in [0,1/N] \quad \text{and} \quad \theta(t) = 0 \quad \text{in a neighborhood of } 2/N. \tag{16}$$

If we denote $\mathcal{L}u = \partial_t u - \Delta u + (u,\nabla)u$, it is very easy to check that

$$\mathcal{L}u_\epsilon = \tilde{f}_\epsilon, \quad (t,x) \in [T-2\delta, T-2\delta+2/N] \times \Omega,$$

for some $\tilde{f}_\epsilon$, together with $u_\epsilon(t,0,x_2) = 0$ for $(t,x_2) \in (T-2\delta, T-2\delta+2/N) \times (0,1)$ and $\nabla \cdot u_\epsilon = 0$ for $(t,x) \in (T-2\delta, T-2\delta+2/N) \times \Omega$.

After some computations and thanks to (10) and (13), one can prove that

$$\|\tilde{f}_\epsilon\|_{L^{p_0}(0,T;V')} \leqslant CN^{7/8-1/p_0}.$$

Thanks to our choice of p_0, this constant tends to zero as $N \to +\infty$.

• Finally, on the interval $[T-2\delta+\frac{2}{N}, T]$, we take $f_\epsilon \equiv 0$ and we try to find a boundary control which drives the associated solution of (5) which starts at time $t = T-2\delta+2/N$ from the initial condition $N^2U(2/N,x)$ to zero at time $t = T$.

Observe that we have $u_\epsilon(T-2\delta+2/N,x) = N^2U(2/N,x)$ since $\theta(2/N) = 0$ and $\tilde{y}(1/N) = 0$ (see (16) and (9)).

Now, from well-known controllability results for the linear heat equation (see, for instance, [5]), for any $\bar{z}_0 \in L^2(0,1)$, there exists a boundary control $\rho = \rho(t) \in L^2(0,2\delta-2/N)$ such that the solution of

$$\begin{cases} \partial_t \bar{z} - \partial^2_{x_1x_1}\bar{z} = 0, & (t,x_1) \in (0,T) \times (0,1), \\ \bar{z}(t,0) = 0, \quad \bar{z}(t,1) = \rho(t), & t \in (0,T), \\ \bar{z}(0,x_1) = \bar{z}_0, & x_1 \in (0,1) \end{cases}$$

satisfies

$$\bar{z}(2\delta-2/N,x_1) = 0, \quad x_1 \in (0,1).$$

Then, it suffices to take

$$u_\epsilon(t,x) = \big(0, \bar{z}(t-T+2\delta-2/N,x_1)\big), \quad (t,x) \in (T-2\delta+2/N, T) \times \Omega,$$

where $\bar{z}$ is the solution of the previous null controllability problem with initial condition

$$\bar{z}_0(x_1) = N^2 z(2/N,x_1), \quad x_1 \in (0,1).$$

This finishes the proof of Theorem 1.1.

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