Abstract

By topological arguments, we set sufficient hypotheses for a given function $K$, on the unit sphere $(S^3, g)$, to be the scalar curvature of a metric conformal to $g$. To cite this article: W. Abdelhedi, C. R. Acad. Sci. Paris, Ser. I 343 (2006).

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Résumé


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1. Introduction and the main results

Let $(S^3, g)$ be the standard 3-sphere equipped with the standard metric. Let $K$ be a $C^2$ positive function on $S^3$. We study the problem:

$$\begin{cases} -8\Delta_g u + 6u = K(x)u^5, \\ u > 0 \text{ on } S^3. \end{cases}$$

Under some conditions on $K$, we prove that this equation has at least one solution.

In this paper, we give a contribution in the spirit of Aubin and Bahri [1] and Bahri and Coron [3], using topology and Bahri’s theory of critical points at infinity (see [2]). The first result here (Theorem 1.1) is that under one qualitative assumption on some of the critical points of $K$ (assumption $(C_1)$) and one topological assumption on the remaining critical points of $K$ (assumption $(C_2)$), then there is a positive solution of (1). This result generalizes, in particular, a result of Bahri and Coron [3] where topological contractibility assumptions on all the critical points of $K$ are assumed (see Corollary 1.2). In Remark 1.5, we describe a situation in which Theorem 1.1 applies, but not Bahri–Coron’s.

In order to state our results, we need to fix some notations and assumptions that we are using.

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Throughout this Note, we assume that $K$ has only non-degenerate critical points $y_0, y_1, \ldots, y_h$ such that $\Delta K(y_i) \neq 0$ for each $i = 0, \ldots, h$ and $K(y_0) \geq K(y_1) \geq \cdots \geq K(y_h)$. Each $y_i$ is assumed to be of index $\text{ind}(K, y_i) = 3 - k_i$. Let $I^+ = \{y_i \mid -\Delta K(y_i) > 0\}$.

Let $Z$ be a pseudo-gradient of $K$ of Morse–Smale type (that is the intersection of stable and unstable manifolds of the critical points of $K$ are transverse). We assume that $W_s(y_i) \cap W_u(y_j) = \emptyset$, for each $y_i \in I^+$ and $y_j \not\in I^+$, where $W_s(y_i)$ is the stable manifold of $y_i$ and $W_u(y_j)$ is the unstable manifold of $y_j$ for $Z$. For each $0 \leq \ell \leq h$, we define $X_\ell = \bigcup_{0 \leq j \leq \ell} \overline{W}_s(y_j)$. We then have:

**Theorem 1.1.** Assume that there exist $\ell \in \{0, \ldots, h\}$ satisfying the following conditions:

1. $K(y_j)^{-1/2} > K(y_0)^{-1/2} + K(y_\ell)^{-1/2}$ for $j \in \{\ell + 1, \ldots, h\}$ and $y_j \in I^+$.
2. $X_\ell$ is not contractible. We denote by $m$ the dimension of the first non-trivial reduced homology group.

Then problem (1) admits a solution.

**Corollary 1.2.** If $\sum_{y_i \in I^+} (-1)^{3 - \text{ind}(K, y_i)} \neq 1$, then (1) has a solution.

**Corollary 1.3.** The solution obtained in Theorem 1.1 has an augmented Morse index $\geq m$.

To state our next result, we need to introduce the following assumptions:

1. There exist $F^+ \subset I^+$ such that $X = \bigcup_{y_i \in F^+} \overline{W}_s(y_i)$ is a stratified set in dimension $k \geq 1$ without boundary (in the topological sense, i.e. $X \in S_k(S^2)$, the group of chains of dimension $k$ and $\partial X = 0$).
2. For all $y \in I^+ \setminus F^+$ we have $\text{ind}(K, y_j) \notin \{3 - k, 3 - (k + 1)\}$.

We then have the following:

**Theorem 1.4.** Under the assumptions (C$_3$) and (C$_4$), the problem (1) admits a solution.

**Remark 1.5.** Here, we give a situation where the result of Corollary 1.2 does not give a solution to problem (1) but by Theorem 1.1 or Theorem 1.4, we derive that problem (1) admits a solution.

For this, let $K : S^3 \to \mathbb{R}$ be a function such that $I^+ = \{y_0, y_1, y_2\}$ with, $K(y_0) \geq K(y_1) \geq K(y_2)$, $\text{ind}(K, y_0) = 3$, $\text{ind}(K, y_1) \neq \text{ind}(K, y_2) \in \{1, 2\}$ and $K(y) < K(y_1)$ for any critical point $y$ of $K$ which is not in $I^+$. It is easy to see that

$$\sum_{y_j \in I^+} (-1)^{3 - \text{ind}(K, y_j)} = 1.$$ 

From another part, $X_1 = \overline{W}_s(y_1) = W_s(y_1) \cup \{y_0\}$ is a stratified set in dimension $\geq 1$, without boundary. Thus, $X_1$ is not contractible. We distinguish two cases:

1. If $K(y_2)^{-1/2} > K(y_0)^{-1/2} + K(y_1)^{-1/2}$, we deduce from Theorem 1.1 that problem (1) has a solution.
2. If $i(y_1) = 1$ and $i(y_2) = 2$, by Theorem 1.4 we derive that (1) has a solution.

2. Proofs of results

Problem (1) is equivalent to finding the critical points of the following function:

$$J(u) = \frac{1}{(\int_{S^2} K(x)u^6 \, dv_g)^{1/3}}, \quad u \in \Sigma^+,$$
where $\Sigma^+ = \{ u \in \Sigma, \ u \geq 0 \}$ and $\Sigma = \{ u \in H^1(S^3), \ |u|^2_{H^1} = 1 \}$. For $a \in S^3, \lambda > 0$, let:

$$\delta(a,\lambda)(x) = c_0 \left( \frac{\lambda}{(\lambda^2 + 1) + (\lambda^2 - 1) \cos d(a,x)} \right)^{1/2},$$

where $\delta(a,\lambda)(x)$ is a solution of the Yamabe problem on $S^3$.

**Proposition 2.1.** (See Lemma 7 of [3]) Assume that $J$ has no critical points in $\Sigma^+$, then the only critical points at infinity for $J$ are $\delta(y_i,\infty)$ such that $y_i \in I^+$, where

$$I^+ = \{ y \in S^3 \mid \nabla K(y) = 0 \text{ and } -\Delta K(y) > 0 \}.$$

The level of such critical point at infinity is $S^{2/3} K(y_i)^{-1/3}$, where $S = \int_{S^3} \delta^3 dv$.

Moreover, the Morse index of the critical point at infinity $\delta(y_i, \infty)$ is given by $i(y_i)_{\infty} = 3 - \text{ind}(K, y_i)$.

**Proof of Theorem 1.1.** Arguing by contradiction, we assume that $J$ has no critical points in $\Sigma^+$. Let $C_{\infty}(y_0, y_\ell) = S^{2/3} \left( \frac{1}{K(y_0)^{1/2}} + \frac{1}{K(y_\ell)^{1/2}} \right)^{2/3}$. It follows from the result of Proposition 2.1 and the assumption (C1) of Theorem 1.1 that, the only critical points at infinity of $J$ under the level $c_1 = C_{\infty}(y_0, y_\ell) + \varepsilon$, for $\varepsilon$ small enough, are $\delta(y_j, \infty)$ where $y_j \in I^+$ and $j \in \{0, \ldots, \ell \}$. In the neighborhood of such critical points at infinity, we have:

$$J(\alpha \delta(a,\lambda) + v) = \frac{S^{2/3}}{K(a)^{1/3}} \left( 1 - \frac{\Delta K(y_j)}{\lambda^2} \right) + |V|^2 \quad \text{(see [1] p. 534)}.$$

In order to define our deformations, we can work as if $V$ was zero. The deformation will extend with the same properties, to a neighborhood of zero in the $V$ part. Thus, the unstable manifolds at infinity for the vector field $(-\partial J)$ of such critical points at infinity $\eta(s,x)$ can be described as the product of $W_s(y_j)$ (for a decreasing pseudo-gradient of $K$) by $[A,\infty]$ domain of the variable $\lambda$, for some positive number $A$ large enough (see [1] p. 535). Since $J$ has no critical points, the set $J_{c_1} = \{ u \in \Sigma^+ \mid J(u) \leq c_1 \}$ retract by deformation onto $(X_{\ell})_{\infty} = \bigcup_{y_j \in I^+} W_u(y_j)$ which can be parameterized by $X_{\ell} \times [A, \infty]$.

Observe that by assumption (C2) of Theorem 1.1 $(X_{\ell})_{\infty}$ is not a contractible set. Now, we prove that $(X_{\ell})_{\infty}$ is contractible in $J_{c_1}$. Indeed, let:

$$f : [0,1] \times (X_{\ell})_{\infty} \rightarrow \Sigma^+$$

$$(t,x,\lambda) \mapsto t\delta(0,\lambda) + (1-t)\delta(x,\lambda).$$

For $t = 0, f(0,x,\lambda) = \frac{1}{2} \delta(x,\lambda) \in X_{\infty}$, $f$ is continuous and $f(1,x,\lambda) = \frac{1}{2} \delta(0,\lambda)$. Let $a_1, a_2 \in S^3, \alpha_1, \alpha_2 > 0$ and $\lambda$ large enough. For $u = \alpha_1 \delta(a_1,\lambda) + \alpha_2 \delta(a_2,\lambda)$, we have:

$$J\left( \frac{u}{|u|_{H^1}} \right) \leq \left( S \left( \frac{1}{K(a_1)^{1/2}} + \frac{1}{K(a_2)^{1/2}} \right) \right)^{2/3} (1 + o(1)),$$

where $o(1) \rightarrow 0$ when $\lambda \rightarrow +\infty$ independently of $t$ and $x$. Hence,

$$J(f(t,x,\lambda)) \leq \left( S \left( \frac{1}{K(y_0)^{1/2}} + \frac{1}{K(y_\ell)^{1/2}} \right) \right)^{2/3} (1 + o(1)).$$

We claim that $K(x) \geq K(y_j)$ for any $x \in X_{\ell}$. Indeed, for each $x \in W_s(y_j)$ we have $\eta(s,x) \rightarrow y_j$ when $s \rightarrow +\infty$, where $\eta(s,x)$ is the decreasing flow of $Z$. Thus, $K(x) \geq K(y_j)$. Furthermore, for $0 \leq j \leq \ell$, we have $K(y_j) \geq K(y_\ell)$ (since $K(y_0) \geq K(y_1) \geq \cdots \geq K(y_\ell)$). Hence, we claim follows and we derive that $J(f(t,x,\lambda)) < c_1$ for any $(t,x,\lambda) \in [0,1] \times X_{\ell} \times [A, \infty]$.

Thus, the contraction $f$ is performed under the level $c_1$. We deduce that, $(X_{\ell})_{\infty}$ is contractible in $J_{c_1}$, which retracts by deformation on $(X_{\ell})_{\infty}$, therefore $(X_{\ell})_{\infty}$ is contractible leading to the contractibility of $X_{\ell}$ which is a contradiction. The proof of Theorem 1.1 is thereby completed. \(\square\)

**Proof of Corollary 1.2.** We recall that $K$ has only non degenerate critical points $y_0, y_1, \ldots, y_h$ such that $K(y_0) \geq K(y_1) \geq \cdots \geq K(y_h)$. For $\ell = h$, we have $X_h = \bigcup_{y_j \in I^+} W_s(y_j)$ then $\chi(X_h) = \sum_{y_j \in I^+} (-1)^{3 - \text{ind}(K,y_j)}$, where
χ(X_h) is the Euler–Poincaré characteristic of X_h (recall that for a stratified set M in dimension l, the Euler–Poincaré characterization of M is given by χ(M) = ∑(-1)^i dim H_i(M), where H_i(M) is the homology group in dimension i associated to M). Under the assumption of Corollary 1.2, we derive that X_h is not contractible. Hence, the result follows from Theorem 1.1. □

Proof of Theorem 1.4. Let X = ∪y_i ∈ F F^+_s(y_i). By the assumption (C3) of Theorem 1.4, X is a stratified set in dimension k ≥ 1 without boundary. For λ large enough, we define the following set, C_λ(y_0, X) = {αδ_{y_0, λ} + (1 − α)δ_{x, λ}, x ∈ X and α ∈ [0, 1]}. C_λ(y_0, X) is a contractible manifold in dimension k + 1, that is its singularities arise in dimension k − 1 and lower.

Let X_∞ = ∪y_i ∈ F F^+_s(y_i)∞. We argue by contradiction, we suppose that J has no critical points in Σ^+. Thus, C_λ(y_0, X) retract by deformation on ∪y ∈ H F^+_s(y)_∞, where H = {y ∈ I^+ | C_λ(y_0, X) ∩ W_s(y)_∞ ≠ ∅}.

Since C_λ(y_0, X) is a manifold in dimension k + 1, this manifold can be assumed to avoid the unstable manifold of every critical point at infinity δ_{y_∞} of Morse index > k + 1, i.e., ind(K, y) < 3 − (k + 1). Thus, H ⊂ {y ∈ F^+ | ind(K, y) ≥ 3 − k}. More precisely, C_λ(y_0, X) retract by deformation on X_∞ ∪ D_∞, where D_∞ = ∪y ∈ D F^+_s(y)_∞ and D = {y ∈ H \ F^+}.

Using the assumption (C4) of Theorem 1.4, we derive that ind(K, y) > 3 − k for each y ∈ D. Thus, the Morse index at infinity of the critical point at infinity δ_{y_∞}, y ∈ D is ≤ k − 1, and therefore D_∞ is a stratified set of dimension at most k − 1. Since C_λ(y_0, X) is a contractible set, then H_k(X_∞ ∪ D_∞) = 0 for all * ∈ N^*. Using the exact homology sequence of (X_∞ ∪ D_∞, X_∞), we have:

\[ \cdots \rightarrow H_{k+1}(X_∞ ∪ D_∞) \rightarrow H_k(X_∞ ∪ D_∞, X_∞) \rightarrow H_k(X_∞) \rightarrow H_k(X_∞ ∪ D_∞) \rightarrow \cdots \]

Since H_k(X_∞ ∪ D_∞) = 0 for all * ∈ N^*, then H_k(X_∞) = H_{k+1}(X_∞ ∪ D_∞, X_∞).

In addition, (X_∞ ∪ D_∞, X_∞) is a stratified set of dimension at most k, so H_{k+1}(X_∞ ∪ D_∞, X_∞) = 0. Thus, H_k(X_∞) = 0 and therefore H_k(X) = 0 which is in contradiction to the assumption (C4) of the theorem. Hence our result follows. □

References