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C. R. Acad. Sci. Paris, Ser. I 343 (2006) 151–154
COMPTES RENDUS MATHEMATIQUE
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Logic

The theory of closed ordered differential fields with m commuting derivations

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Received 6 June 2006; accepted 13 June 2006

Available online 7 July 2006

Presented by Jean-Yves Girard

Abstract

We generalize the work of M. Singer (1978) on the theory of closed ordered differential fields to the case of m-ODF, the theory of ordered fields equipped with m commuting derivations. We give an algebraic axiomatization of the model completion (denoted by m-CODF) of this theory and we can immediately deduce that m-CODF has quantifier elimination in the natural language of ordered Δ-rings. ***To cite this article: C. Rivière, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

La théorie des corps ordonnés différentiellement clos munis de *m* dérivations commutant entre elles. Nous généralisons les travaux de M. Singer concernant la théorie des corps ordonnés différentiellement clos au cas des corps ordonnés munis de m dérivations commutant entre elles. Nous donnons une axiomatisation algébrique de la modèle-complétion de cette théorie et nous pouvons directement déduire que cette dernière admet l'élimination des quantificateurs dans le langage naturel des anneaux ordonnés différentiels. ***Pour citer cet article : C. Rivière, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Basic differential algebra

A *Δ-ring* (*resp. Δ-field*) is a ring (resp. field) M equipped with a set $\Delta = \{\delta_1, \ldots, \delta_m\}$ of m commuting derivations (e.g. the field $\mathbb{R}(X, Y)$ of rationals functions over $\mathbb{R}$ equipped with the usual partial derivations w.r.t. X and Y is a differential field).

Let $a \in M$ where M is a Δ-field of characteristic zero, we use the notation ${\delta_1}^{(e_1)} \cdots {\delta_m}^{(e_m)} a$ to denote the element $\underbrace{\delta_1 \cdots \delta_1}_{e_1 \text{ times}} \cdots \underbrace{\delta_m \cdots \delta_m}_{e_m \text{ times}} a$ of M.

An ideal I of M is a *Δ-ideal* if it is closed under the action of Δ. For any subset S of M, we write (S) for the ideal generated by S in M and $[S]$ for the Δ-ideal generated by S in M.

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[1] The author was supported by a FRIA grant during his PhD thesis at the University of Mons-Hainaut and is now supported by a MODNET grant.

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doi:10.1016/j.crma.2006.06.019

Let Θ be the set of derivative operators $\{\delta_1{}^{(e_1)} \cdots \delta_m{}^{(e_m)} \mid e_1, \ldots, e_m \geqslant 0\}$ then $M\{y\}$ is the polynomial ring generated by the θy's and is called the *ring of Δ-polynomials* in 1 indeterminate over M. Remark that the derivations $\delta_1, \ldots, \delta_m$ extend naturally to $M\{y\}$ by putting

$$\delta_i(\theta y) = \delta_1{}^{(e_1)} \cdots \delta_{i-1}{}^{(e_{i-1})} \delta_i{}^{(e_i+1)} \delta_{i+1}{}^{(e_{i+1})} \cdots \delta_m{}^{(e_m)} y \quad \text{if } \theta = \delta_1{}^{(e_1)} \cdots \delta_m{}^{(e_m)}.$$

We define a *ranking* on Θ by setting $\delta_1{}^{(e_1)} \cdots \delta_m{}^{(e_m)} < \delta_1{}^{(e'_1)} \cdots \delta_m{}^{(e'_m)}$ iff $(e, e_m, \ldots, e_1) < (e', e'_m, \ldots, e'_1)$ in the lexicographical ordering (where $e := \sum_{i=1}^m e_i$ and $e' := \sum_{i=1}^m e'_i$ are called the *order* of respectively $\delta_1{}^{(e_1)} \cdots \delta_m{}^{(e_m)}$ and $\delta_1{}^{(e'_1)} \cdots \delta_m{}^{(e'_m)}$). We will denote by θ_{h-1} the h-th element of Θ w.r.t. this ranking. Remark that the order type of this ranking is ω and that if $\theta_1 < \theta_2$ and θ are in Θ then $\theta\theta_1 < \theta\theta_2$.

Let f be in $M\{y\}$, the maximal h such that $\theta_h y$ appears non trivially in f is called the *height* of f and is denoted by h_f. Furthermore, the *order* of f (denoted by $ord(f)$) is equal to the order of θ_{h_f} (remark that this definition only holds in the case of Δ-polynomials in one single indeterminate, see [1]) and the *leader* of f (denoted by v_f) is $\theta_{h_f} y$.

We associate two Δ-polynomials to f: the *separant* of f (denoted by $S_f(y)$) is the partial derivative of f with respect to v_f and the *initial* of f (denoted by $I_f(y)$) is the leading coefficient of f considered as an ordinary polynomial in the variable v_f. We also define the *rank* of a Δ-polynomial f to be the lexicographically ordered pair (h_f, d_f) where d_f is the degree of f considered as a polynomial in v_f.

Definition 1.1. Let $f_1, f_2 \in M\{y\}$, we say that f_1 is *partially reduced w.r.t.* f_2 if no proper derivative of v_{f_2} appears (non trivially) in f_1. If furthermore $deg_{v_{f_2}}(f_1) < deg_{v_{f_2}}(f_2)$ (where we consider f_1, f_2 as ordinary polynomials in v_{f_2}) then we say that f_1 is *reduced w.r.t.* f_2.

A subset $F = \{f_1, \ldots, f_s\}$ of $M\{y\}$ is *autoreduced* if, for any $i \neq j$, f_i is reduced w.r.t. f_j.

Remark that if f_i, f_j are reduced w.r.t. each other then $v_{f_i} \neq v_{f_j}$ and we can always assume that in an autoreduced set $F = \{f_1, \ldots, f_s\}$ the Δ-polynomials are ranked in order of increasing height.

Let $F = \{f_1, \ldots, f_s\}$ be an autoreduced set of Δ-polynomials, then we define the following Δ-polynomial $H_F := \prod_{i=1}^s I_{f_i} S_{f_i}$. Remark that, since F is autoreduced, H_F is partially reduced w.r.t. F.

Definition 1.2. An autoreduced set $F = \{f_1, \ldots, f_s\} \subseteq M\{y\}$ is *coherent* if for any $i \neq j$, if θ_h is the least (in the ranking of Θ) derivative operator such that there exist $\theta_i, \theta_j \in \Theta$ with $\theta_i v_{f_i} = \theta_j v_{f_j} = \theta_h y$ then $S_{f_j}\theta_i f_i - S_{f_i}\theta_j f_j$ belongs to $(F)_{h-1}$ which is the ideal of $M\{y\}$ generated by the θf_i with $\theta\theta_{h_{f_i}} \leqslant \theta_{h-1}$ (remark that $[F] = \bigcup_{h\in\mathbb{N}} (F)_h$).

2. Axiomatization of *m*-CODF

We now consider an ordered Δ-field M, i.e. an ordered field equipped with a set Δ of m commuting derivations which do not interact with the order.

Definition 2.1. M is a *closed ordered Δ-field* if it is real closed and, for any coherent autoreduced set $F = \{f_1, \ldots, f_s\} \subset M\{y\}$ such that the ideal $(F) : H_F{}^\infty := \{f \in M\{y\} \mid H_F{}^n f \in (F) \text{ for some } n \in \mathbb{N}\}$ is prime and does not contain any nonzero element reduced w.r.t. F, and any $g_1, \ldots, g_l \in M\{y\}$ reduced w.r.t. F, the system

$$\left(\bigwedge_{i=1}^{s} f_i(y) = 0 \wedge H_F(y) \neq 0 \wedge \bigwedge_{j=1}^{l} g_j(y) > 0 \right) \tag{$*$}$$

has a differential solution as soon as the system $(\tilde{*})$, obtained from $(*)$ by considering the Δ-polynomials as ordinary polynomials (i.e. we replace any $\theta_i y$ appearing in $(*)$ by a new variable X_i) has an algebraic solution $(x_0, \ldots, x_r)$ in M.

We denote by m-CODF the theory of closed ordered Δ-fields in the language $L^{\Delta}_{<} = \{+, -, *, \delta_1, \ldots, \delta_m, 0, 1, <\}$.

The axioms in Definition 2.1 can be proved to be first-order (in the coefficients of $f_1, \ldots, f_s$) using the work in [4] and the fact that $(F) : H_F{}^\infty = ((F), XH_F - 1) \cap M\{y\}$ where X is a new indeterminate. Details can be found in [3, Chapter 4].

From now on, we write $\tilde{f}$ for the polynomial in the variables $X_0, \ldots, X_{h_f}$ obtained from f by replacing any $\theta_l y$ by a new variable X_l ($l \leqslant h_f$) and $\tilde{F}$ for the set of polynomials $\{\tilde{f}_1, \ldots, \tilde{f}_s\}$.

Theorem 2.2. *The theory m-CODF of closed ordered Δ-fields is the model completion of the theory m-ODD of ordered Δ-domains (in particular it is the model completion of m-ODF).*

Furthermore, since m-ODD is universally axiomatized in $L_<^\Delta$, m-CODF has quantifier elimination in this language.

To prove this theorem we have to show first that each ordered Δ-field extends to a model of m-CODF and then that we can complete any diagram as in Blum's criterion (see [2, Theorem 17.2]).

Proof (1). Let M be an ordered Δ-domain. Since the derivations and the order on M extend uniquely to the real closure of the quotient field of M, we can assume that $M \models m$-ODF and is a real closed field.

Let $f_1, \ldots, f_s, g_1, \ldots, g_l \in M\{y\}$ be as in the axioms of m-CODF and remark first that the fact that $(F) : {H_F}^\infty$ is prime and does not contains any nonzero element reduced w.r.t. F implies that the Δ-polynomials $f_1, \ldots, f_s$ are irreducible.

Assume that h_i is the height of f_i (with $h_1 < \cdots < h_s$) and that h_r is maximal amongst the heights of the elements of $F \cup \{g_1, \ldots, g_l\}$. We consider the set I of positive integers n such that $\theta_n(y)$ appears non trivially in one of the Δ-polynomials in $F \cup \{g_1, \ldots, g_l\}$ and denote by J the set $\{h_1, \ldots, h_s\}$ (obviously, $J \subseteq I$). Furthermore for any $i \in \{1, \ldots, s\}$ we define $D_i := \{n \in \mathbb{N} \mid \exists j \geqslant 1\ \theta_n(y) = \theta_j(v_{f_i})\}$ and $D := \bigcup_{i=1}^s D_i$.

Remark that, since F is autoreduced and $\{g_1, \ldots, g_l\}$ is reduced w.r.t. F, $D \cap I = \emptyset$.

We now consider an infinite tuple $\bar{a} = (a_0, a_1, \ldots) \in M^\omega$ which is a solution of the system $(\tilde{*})$.

By [1, Lemma IV.9.2] the ideal $P = [F] : {H_F}^\infty$ is a prime Δ-ideal with characteristic set F. Moreover if $0 \neq g \in M\{y\}$ is reduced with respect to F, then $g(y) \notin P$ (see [1, IV.9.2] and also the remark following [1, Lemma III.2.1]). Let L be the field of fractions of $M\{y\} \setminus P$, and denote the image of y in L by c. Note that the derivations in Δ extend uniquely to L and also to its algebraic closure.

We will show that we can define an ordering on L which extends the ordering on M and satisfies: for each $i \in I$ the element $\theta_i c - a_i$ is infinitesimal with respect to M. This will imply that c is a solution of $(*)$, since each $g_j(c) - \widetilde{g_j}(\bar{a})$ will then be infinitesimal with respect to M, and therefore $g_j(c)$ will have the same sign as $\widetilde{g_j}(\bar{a})$.

We will define recursively the ordering on each field $L_i := M(c, \ldots, \theta_i c)$ for $i \in \mathbb{N}$.

Case 1. $i \notin D \cup J$.

If $f(y) \in M\{y\}$ is of height i then $f(y)$ is reduced with respect to F. Hence such an $f(y)$ does not belong to P. Thus $\theta_i c$ is transcendental over L_{i-1}. We can therefore extend the ordering of L_{i-1} to L_i so that $\theta_i c - a_i$ is infinitesimal with respect to L_{i-1}.

Case 2. $i \in J$.

Then there exists $j \in \{1, \ldots, s\}$ with $i = h_j$. Remark that, since $H_F(\bar{a}) \neq 0$, a_i is a simple root of the polynomial $\tilde{f}_j(a_0, \ldots, a_{i-1}, X_i)$ and that the coefficients of this polynomial are infinitesimally close to those of $f_j(c, \ldots, \theta_{i-1}c, X_i)$. Hence, since polynomial functions are continuous for the order topology, these two polynomials have the same degree in X_i and $\tilde{f}_j(c, \ldots, \theta_{i-1}c, X_i)$ has a simple root d in the real closure of L_{i-1}.

Case 3. $i \in D$.

Assume first that $\theta_i y = \delta_u \theta_{h_j}$ for some $j \in \{1, \ldots, s\}$. Using the fact that $f_j(c) = 0$ and $S_{f_j}(c) \neq 0$, and that if $h < h_j$, then $\delta_u \theta_h < \theta_i$, we obtain that $\theta_i c \in L_{i-1}$. In the general case, $\theta_i y = \theta\theta_h$ for some $j \in \{1, \ldots, s\}$ and $\theta \in \Theta$, and an easy induction on the order of θ shows that $\theta_i c \in L_{i-1}$.

Using a transfinite induction one can build an ordered differential field which satisfies the axioms of m-CODF.

Proof (2). We want to check that Blum's criterion holds (as before we can assume that M and $M(a)$ are models of m-ODF). For this, let M^* be an $|M|^+$-saturated elementary extension of M and a an element in some ordered Δ-field extending M.

(a) Suppose first that a is Δ-algebraic over M, i.e. there exists a Δ-polynomial $f \in M\{y\}$ such that $f(a) = 0$.

Let I be the prime Δ-ideal $I\langle a/M\rangle = \{f \in M\{y\} \mid f(a) = 0\}$. By [1, Proposition 3 p. 81 and Lemma 2 p. 167], there exists an autoreduced coherent subset $F = \{f_1, \ldots, f_s\}$ of I such that $(F) : {H_F}^\infty$ is prime, contains no nonzero reduced element w.r.t. F, and $I = [F] : {H_F}^\infty$. Then the isomorphism type of the ordered Δ-field $M\langle a\rangle$ generated by a over M is completely determined by the equations $f_1(a) = \cdots = f_s(a) = 0$ and a list of inequations of the form $g_j(a) > 0$ where g_j is a Δ-polynomial that we can assume to be reduced w.r.t. F by [1, Proposition 1, p. 79]. Since I_{f_j} and S_{f_j} are reduced w.r.t. F for any $j \in \{1, \ldots, s\}$ and $(F) : {H_F}^\infty$ is prime, H_F does not belong to this ideal. It follows, by [1, Lemma 5 p. 137], that H_F does not belong to the Δ-ideal $[F] : {H_F}^\infty$.

Hence, for any system $\mathcal{S} \equiv (\bigwedge_{i=1}^{s} f_i(y) = 0 \wedge H_F(y) \neq 0 \wedge \bigwedge_{j=1}^{l} g_j(y) > 0)$ where $g_1, \ldots, g_l \in M\{y\}$ is a finite collection of Δ-polynomials reduced w.r.t. F such that $g_j(a) > 0$, there exists an algebraic solution to the system $\tilde{\mathcal{S}}$ in an ordered field extending M (namely, this solution is $(a, \theta_1(a), \ldots, \theta_r(a))$ where r is the maximal height of an element of $F \cup \{g_1, \ldots, g_l\}$).

Since M^* is a real closed field, it also contains an algebraic solution of $\tilde{\mathcal{S}}$. Furthermore $M^* \models m$-CODF and hence $\mathcal{S}$ has a differential solution u in M^*. By the saturation of M^* there exists a solution c in M^* to all the above systems where the g_j's range over the Δ-polynomials reduced w.r.t. F. In other words $M\langle c\rangle$ is $(L^{\Delta}_{<})$-isomorphic to $M\langle a\rangle$.

(b) The case when a is not Δ-algebraic (i.e. is Δ-transcendental) over M can be proved similarly: consider systems $\mathcal{S} \equiv (\theta_{h_r+1}(y) = 0 \wedge \bigwedge_{j=1}^{l} g_j(y) > 0)$ where $g_1, \ldots, g_l$ have height at most h_r. Letting h_r tend to ∞, the axiomatization of m-CODF and the saturation of M^* provide an element c in M^* which is Δ-transcendental over M. In other words c is such that $M\langle c\rangle$ is $(L^{\Delta}_{<})$-isomorphic to $M\langle a\rangle$.

Acknowledgements

The result proved here was part of my PhD thesis, I would like to thank my advisor Christian Michaux for his precious advice during the writing of this work. Several previous versions of this text contained serious gaps in the point (1) of the proof of Theorem 2.2, I thank the referee for his patience, his help in the resolution of these problems and his indications to shorten this proof.

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