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Group Theory

A refinement of Harish-Chandra's method of descent

Florent Bernon

Département de mathématiques, université Paris X-Nanterre, 200, avenue de la République, 92000 Nanterre, France

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Abstract

Let G be a connected real reductive group and M a connected reductive subgroup of G with Lie algebras $\mathfrak{g}$ and $\mathfrak{m}$ respectively. We assume that $\mathfrak{g}$ and $\mathfrak{m}$ have the same rank. We define a map from the space of orbital integrals of $\mathfrak{m}$ into the space of orbital integrals of $\mathfrak{g}$ which we call a transfer. The transpose of the transfer can be viewed as a map from the space of G-invariant distributions of $\mathfrak{g}$ to the space of M-invariant distributions of $\mathfrak{m}$ and can be considered as a restriction map from $\mathfrak{g}$ to $\mathfrak{m}$. We prove that this restriction map extends Harish Chandra's method of descent and we obtain a generalization of Harish-Chandra's radial component theorem. ***To cite this article: F. Bernon, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Une extension de la méthode de descente de Harish-Chandra. Soient G un groupe réductif réel connexe et M un sous-groupe réductif connexe de G d'algèbres de Lie respectivement $\mathfrak{g}$ et $\mathfrak{m}$. On suppose que $\mathfrak{g}$ et $\mathfrak{m}$ ont le même rang. Nous prouvons qu'il existe une application de l'espace des intégrales orbitales de $\mathfrak{m}$ dans l'espace des intégrales orbitales de $\mathfrak{g}$ que l'on appelle un transfert. La transposée de ce transfert définit une application de l'espace des distributions G-invariante sur $\mathfrak{g}$ dans l'espace des distributions M-invariantes sur $\mathfrak{m}$ et peut être considérée comme une restriction. On montre que cette application de restriction étend la méthode de descente de Harish-Chandra et on obtient une généralisation du théorème de la composante radiale de Harish-Chandra. ***Pour citer cet article : F. Bernon, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

1. Introduction

Let G be a connected reductive group and M a connected reductive subgroup of G (the class of groups considered is defined later) with Lie algebras $\mathfrak{g}$ and $\mathfrak{m}$ respectively. We throughout assume that $\mathfrak{g}$ and $\mathfrak{m}$ have the same rank. We denote by $\mathrm{Car}(\mathfrak{g})$ (resp. $\mathrm{Car}(\mathfrak{m})$) the set of Cartan subalgebras of $\mathfrak{g}$ (resp. $\mathfrak{m}$). Then, $\mathrm{Car}(\mathfrak{m}) \subset \mathrm{Car}(\mathfrak{g})$. Consider the adjoint action of G on $\mathfrak{g}$ and of M on $\mathfrak{m}$. Notice first that for a semisimple regular element x of $\mathfrak{g}$, $\mathrm{G}.x \cap \mathfrak{m}$ is a finite union of M-orbits

$$\mathrm{G}.x \cap \mathfrak{m} = \coprod_{1 \leqslant i \leqslant p} \mathrm{M}.x_i. \tag{1}$$

E-mail address: florent.bernon@u-paris10.fr (F. Bernon).

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Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{m}$ and $\Sigma_{\mathfrak{g}}$ a set of positive roots of the root system $\Phi_{\mathfrak{g}}$ of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Let $\Sigma_{\mathfrak{m}} = \Sigma \cap \Phi_{\mathfrak{m}}$ where $\Phi_{\mathfrak{m}}$ is the root system of $(\mathfrak{m}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Here we write $\mathfrak{h}_{\mathbb{C}}$ (resp. $\mathfrak{m}_{\mathbb{C}}$) for the complexification of $\mathfrak{h}$ (resp. $\mathfrak{m}$) and $G_{\mathbb{C}}$ (resp. $M_{\mathbb{C}}$) for the adjoint group of $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{m}_{\mathbb{C}}$). Fix a non degenerate $G_{\mathbb{C}}$-invariant bilinear form k on $\mathfrak{g}_{\mathbb{C}}$. For $x \in \mathfrak{m}$, the endomorphism $\mathrm{ad}(x)$ is antisymmetric with respect to k therefore we can consider the following Pfaffian:

$$\pi_{\mathfrak{g}/\mathfrak{m}}(x) = \mathrm{Pfaff}\big(\mathrm{ad}(x)_{\mathfrak{g}_{\mathbb{C}}/\mathfrak{m}_{\mathbb{C}}}\big),$$

for $x \in \mathfrak{m}_{\mathbb{C}}$. This polynomial function is $M_{\mathbb{C}}$-invariant. We denote by $\mathcal{D}_{\mathfrak{g}}$ and $\mathcal{D}_{\mathfrak{m}}$ the Weyl denominators on $\mathfrak{g}$ and $\mathfrak{m}$ respectively. Note that

$$|\pi_{\mathfrak{g}/\mathfrak{m}}| = \frac{\mathcal{D}_{\mathfrak{g}}}{\mathcal{D}_{\mathfrak{m}}}.$$

Let $\mathfrak{g}^{\mathrm{reg}}$ be the set of regular semisimple elements of $\mathfrak{g}$. Let $\mathfrak{F}(\mathfrak{g})$ (resp. $\mathfrak{F}(\mathfrak{m})$) be the space of G-invariant functions (resp. M-invariant functions) f, smooth on $\mathfrak{g}^{\mathrm{reg}}$ (resp. on $\mathfrak{g}^{\mathrm{reg}} \cap \mathfrak{m}$) such that $\mathcal{D}_{\mathfrak{g}} f$ (resp. $\mathcal{D}_{\mathfrak{m}} f$) is bounded.

Let $\mathfrak{a}$ be a subspace of $\mathfrak{g}$ such that the restriction of k to $\mathfrak{a} \times \mathfrak{a}$ is non degenerate. Then we consider on $\mathfrak{a}$ the Lebesgue measure da attached to $\mathsf{k}_{|\mathfrak{a}\times\mathfrak{a}}$. If A is a Lie subgroup of G such that $\mathfrak{a}$ is the Lie algebra of A then we consider on A the Haar measure tangential to da. Note that the subspace $\mathfrak{a}$ can in particular be the Lie subalgebra $\mathfrak{m}$.

We denote by $\mathcal{D}(\mathfrak{g})$ the space of test functions on $\mathfrak{g}$. We consider the space of functions

$$\mathfrak{g}^{\mathrm{reg}} \ni x \longmapsto \int_{G/H} \phi(g.x)\,\mathrm{d}\dot{g}$$

where $\phi \in \mathcal{D}(\mathfrak{g})$, H is the Cartan subgroup of G such that x belongs to the Lie algebra of H and $d\dot{g}$ is a quotient measure on G/H, dg/dh. This space is called the space of orbital integrals of $\mathfrak{g}$ and is denoted by $\mathcal{I}(\mathfrak{g})$. We know from Harish-Chandra that $\mathcal{I}(\mathfrak{g})$ is a subspace of $\mathfrak{F}(\mathfrak{g})$. We consider also the space $\mathcal{I}(\mathfrak{m})$ of orbital integrals of $\mathfrak{m}$. As the space $\mathfrak{m} \cap \mathfrak{g}^{\mathrm{reg}}$ is a dense subspace of $\mathfrak{m}^{\mathrm{reg}}$ and the orbital integrals on $\mathfrak{m}$ are smooth on $\mathfrak{m}^{\mathrm{reg}}$, we can consider the space $\mathcal{I}(\mathfrak{m})$ as a subspace of $\mathfrak{F}(\mathfrak{m})$.

The space $\mathcal{I}(\mathfrak{g})$ is well known and is characterized (see [1, Theorem 4.1.1]). We define a transfer from the space $\mathfrak{F}(\mathfrak{m})$ to $\mathfrak{F}(\mathfrak{g})$ by:

$$m_1 : \mathfrak{F}(\mathfrak{m}) \longrightarrow \mathfrak{F}(\mathfrak{g})$$
$$\psi \longmapsto \phi$$

where $\phi(x) = \sum_{1\leqslant i\leqslant p}((\pi_{\mathfrak{g}/\mathfrak{m}})^{-1}\psi)(x_i)$ (see Eq. (1)). We can also consider a restriction map from $\mathfrak{F}(\mathfrak{g})$ to $\mathfrak{F}(\mathfrak{m})$:

$$m_2 : \mathfrak{F}(\mathfrak{g}) \longrightarrow \mathfrak{F}(\mathfrak{m})$$
$$\phi \longmapsto \overline{\pi_{\mathfrak{g}/\mathfrak{m}}}\phi_{|\mathfrak{m}\cap\mathfrak{g}^{\mathrm{reg}}}.$$

If M has only one class of Cartan subgroups then it can easily be proved that $m_1(\mathcal{I}(\mathfrak{m})) \subset \mathcal{I}(\mathfrak{g})$. The main result of this paper is the following: We prove that the restriction of the map m_1 to $\mathcal{I}(\mathfrak{m})$ takes values in $\mathcal{I}(\mathfrak{g})$. We denote by $\mathrm{Tr}_{\mathfrak{g}/\mathfrak{m}}$ the restriction of m_1 to $\mathcal{I}(\mathfrak{m})$ and we say that the transfer $(\mathrm{Tr}_{\mathfrak{g}/\mathfrak{m}})$ is *defined*.

We denote by $\mathcal{D}'^{,G}(\mathfrak{g})$ and $\mathcal{D}'^{,M}(\mathfrak{m})$ the space of G-invariant distributions on $\mathfrak{g}$ and M-invariant distributions on $\mathfrak{m}$. We can consider $\mathfrak{F}(\mathfrak{g})$ as a subspace of $\mathcal{D}'^{,G}(\mathfrak{g})$. Then, we prove that the map m_2 extends to a map from $\mathcal{D}'^{,G}(\mathfrak{g})$ to $\mathcal{D}'^{,M}(\mathfrak{m})$ (Theorem 2.2). We denote this map by $\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}$.

Harish-Chandra introduced a method of descent in Harmonic Analysis (see [2, Introduction]). Let x be a semisimple element of $\mathfrak{g}$. Put $\mathfrak{m} = \mathfrak{g}^x$ and $M = G^x$. Let $\mathcal{V}$ be an open M-invariant neighborhood of x in $\mathfrak{m}$. We assume that $\mathcal{V}$ is G-admissible (see [2, p. 27] or [1, Section 2] for the definition). Then, $\mathcal{U} = G.\mathcal{V}$ is a G-invariant open neighborhood of x in $\mathfrak{g}$. Harish-Chandra proved that we can restrict any G-invariant distributions d defined on $\mathcal{U}$ to $\mathcal{V}$ ($d_{|\mathcal{V}}$) and this map is a bijection. We prove that the maps $\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}$ and $_{|\mathcal{V}}$ coincide up to a known constant.

We prove also a generalization of Harish-Chandra's radial component theorem (Theorem 2.3).

2. Main results

Let G be a real reductive Lie group with Lie algebra $\mathfrak{g}$. Write $G_{\mathbb{C}}$ for the complex adjoint group of $\mathfrak{g}_{\mathbb{C}}$. We say that G belongs to the class $\hat{\mathcal{H}}$ if

i. G has a finite number of connected components.
ii. Ad(G) is connected and $\mathrm{Ad(G)} \subset \mathrm{G}_{\mathbb{C}}$.
iii. The connected subgroup of G with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ has a finite center.

The class $\widehat{\mathcal{H}}$ is contained in the class of Harish-Chandra. Throughout, G will denote a group in the class $\widehat{\mathcal{H}}$ and M a reductive subgroup of G in the class $\widehat{\mathcal{H}}$ and we assume that G and M have the same rank.

The main result of this Note is the following:

Theorem 2.1. *We have* $m_1(\mathcal{I}(\mathfrak{m})) \subset \mathcal{I}(\mathfrak{g})$ *hence the transfer* $\mathrm{Tr}_{\mathfrak{g}/\mathfrak{m}}$ *is defined.*

We denote by $\mathcal{D}'^{,\mathrm{G}}(\mathfrak{g})$ (resp. $\mathcal{D}'^{,\mathrm{M}}(\mathfrak{m})$) the space of G-invariant distributions of $\mathfrak{g}$ (M-invariant distributions of $\mathfrak{m}$). Considering the canonical isomorphisms $\mathcal{I}(\mathfrak{m})' \simeq \mathcal{D}'^{,\mathrm{M}}(\mathfrak{m})$ and $\mathcal{I}(\mathfrak{g})' \simeq \mathcal{D}'^{,\mathrm{G}}(\mathfrak{g})$ (see Theorem 4.1.1 of [1]), we see that the transpose of $\mathrm{Tr}_{\mathfrak{g}/\mathfrak{m}}$ induces a linear map

$$\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}} : \mathcal{D}'^{,\mathrm{G}}(\mathfrak{g}) \longrightarrow \mathcal{D}'^{,\mathrm{M}}(\mathfrak{m}).$$

Notice that we have the canonical injections:

$$\mathfrak{F}(\mathfrak{g}) \hookrightarrow \mathcal{D}'^{,\mathrm{G}}(\mathfrak{g}) \quad \text{and} \quad \mathfrak{F}(\mathfrak{m}) \hookrightarrow \mathcal{D}'^{,\mathrm{M}}(\mathfrak{m}).$$

Theorem 2.2. *The morphism* $\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}$ *maps* $\mathfrak{F}(\mathfrak{g})$ *into* $\mathfrak{F}(\mathfrak{m})$*, and for* $\phi \in \mathfrak{F}(\mathfrak{g})$*,*

$$\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}(\phi) = \overline{\pi_{\mathfrak{g}/\mathfrak{m}}} \phi_{|\mathfrak{m}}.$$

We want now to describe the action of the G-invariant differential operators with constant coefficients on the map $\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}$.

Let V be a real finite dimensional vector space and let A be a group acting on $V_{\mathbb{C}}$. We denote by $\mathrm{Sym}(V_{\mathbb{C}})$ the symmetric algebra of $V_{\mathbb{C}}$ and by $\mathrm{Sym}^{\mathrm{A}}(V_{\mathbb{C}})$ the subspace of A-invariant element of $\mathrm{Sym}(V_{\mathbb{C}})$. Let $\mathrm{D}(V_{\mathbb{C}})$ be the space of differential operators with constant coefficients on V and $\mathrm{D}^{\mathrm{A}}(V_{\mathbb{C}})$ the space of A-invariant differential operators of $\mathrm{D}(V_{\mathbb{C}})$. We consider an isomorphism of algebras between $\mathrm{Sym}(V_{\mathbb{C}})$ and $\mathrm{D}(V_{\mathbb{C}})$ given by

$$\partial(u) f(x) = \lim_{t \to 0} t^{-1}\big(f(x+tu) - f(x)\big),$$

for $u \in V_{\mathbb{C}}$. This induces an isomorphism between $\mathrm{Sym}^{\mathrm{A}}(V_{\mathbb{C}})$ and $\mathrm{D}^{\mathrm{A}}(V_{\mathbb{C}})$. Fix an A-invariant Lebesgue measure on V. On $\mathrm{D}(V_{\mathbb{C}})$, we consider the involution $w \longmapsto w^t$ such that

$$\int_V f(x) \partial(w) g(x)\, \mathrm{d}x = \int_V \partial\big(w^t\big) f(x) g(x)\, \mathrm{d}x$$

for any $f, g \in \mathcal{D}(V)$. This involution stabilizes $\mathrm{D}^A(V)$.

Let $\mathcal{D}'(V)$ denote the space of distributions on V. For $u \in \mathcal{D}'(V)$ and $w \in \mathrm{Sym}(V_{\mathbb{C}})$, we put

$$\big\langle \partial(w) u, \phi \big\rangle = \big\langle u, \partial\big(w^t\big)\phi \big\rangle,$$

for $\phi \in \mathcal{D}(V)$.

There exists a canonical projection map

$$\mathrm{Sym}(\mathfrak{g}_{\mathbb{C}}) \longrightarrow \mathrm{Sym}(\mathfrak{m}_{\mathbb{C}}) w \longmapsto w_{|\mathfrak{m}_{\mathbb{C}}}.$$

This is an algebra homomorphism and for $w \in \mathrm{Sym}^{\mathrm{G}_{\mathbb{C}}}(\mathfrak{g}_{\mathbb{C}})$, we have $w_{|\mathfrak{m}_{\mathbb{C}}} \in \mathrm{Sym}^{\mathrm{M}_{\mathbb{C}}}(\mathfrak{m}_{\mathbb{C}})$. The following theorem can be viewed as a generalization of Harish-Chandra's radial component theorem:

Theorem 2.3. *Let* $w \in \mathrm{Sym}^{\mathrm{G}_{\mathbb{C}}}(\mathfrak{g}_{\mathbb{C}})$ *and* $u \in \mathcal{D}'^{,\mathrm{G}}(\mathfrak{g})$*. We have*

$$\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}\big(\partial(w) u\big) = \partial(w_{|\mathfrak{m}_{\mathbb{C}}})\, \mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}(u).$$

3. Restriction of some fundamental invariant measures

Let $x \in \mathfrak{g}^{\mathrm{reg}}$ (resp. $x \in \mathfrak{m}^{\mathrm{reg}}$), we denote by $\mu_{\mathfrak{g},x}$(resp. $\mu_{\mathfrak{m},x}$) the Borel measure supported on the orbit G.x (resp. M.x). This is a G-invariant distribution (resp. M-invariant distribution). We normalize the measures as we did for orbital integrals.

Theorem 3.1. *Let $x \in \mathfrak{g}^{\mathrm{reg}}$. Then, we have*

$$\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}(\mu_{\mathfrak{g},x}) = \sum_i \pi_{\mathfrak{g}/\mathfrak{m}}(x_i)\mu_{\mathfrak{m},x_i},$$

where $x_1, \ldots, x_p$ satisfy (1).

Let $\delta_{\mathfrak{g}}$ (resp. $\delta_{\mathfrak{m}}$) be the Dirac measure at the origin on $\mathfrak{g}$ (resp. $\mathfrak{m}$). For $\mathfrak{h} \in \mathrm{Car}(\mathfrak{g})$, we denote by $W_{\mathrm{G}}(\mathfrak{h})$ the Weyl group of $\mathfrak{h}$ in G. The same for $W_{\mathrm{M}}(\mathfrak{h})$ with $\mathfrak{h} \in \mathrm{Car}(\mathfrak{m})$. Let K be a maximal compact subgroup of G and $\mathrm{K_M}$ a maximal compact subgroup of M such that $\mathrm{K_M} \subset \mathrm{K}$.

Theorem 3.2. *If $\mathfrak{g}$ and $\mathfrak{m}$ share a common fundamental Cartan subalgebra $\mathfrak{h}$, then we have*

$$\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}(\delta_{\mathfrak{g}}) = w_{\mathfrak{g}/\mathfrak{m}} \frac{|W_{\mathrm{G}}(\mathfrak{h})|}{|W_{\mathrm{M}}(\mathfrak{h})|} \partial(\pi_{\mathfrak{g}/\mathfrak{m}})\delta_{\mathfrak{m}}$$

where $w_{\mathfrak{g}/\mathfrak{m}} = (-1)^{\frac{1}{2}(\dim(\mathrm{G/M}) - \dim(\mathrm{K/K_M}))}(2\pi)^{\frac{1}{2}\dim(\mathrm{G/M})}$. *Otherwise, we have* $\mathrm{Res}_{\mathfrak{g}/\mathfrak{m}}(\delta_{\mathfrak{g}}) = 0$.

This result is a direct consequence of [3, Theorem 8.4.5.1] and Theorem 3.1.

In a forthcoming paper, we will prove these results and give some applications.

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