Probability Theory

Asymptotics for the distribution of lengths of excursions of a $d$-dimensional Bessel process $(0 < d < 2)$

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Abstract

Let $(R_t, t \geq 0)$ denote a $d$-dimensional Bessel process $(0 < d < 2)$. For every $t \geq 0$, we consider the times $g_t = \sup\{s \leq t: R_s = 0\}$, and $d_t = \inf\{s > t: R_s = 0\}$, as well as the three sequences: $(V^n_{g_t}, n \geq 1)$, $(V^n_{d_t}, n \geq 2)$, and $(V^n_{d_t}, n \geq 2)$, which consist of the lengths of excursions of $R$ away from 0 before $g_t$, before $t$, and before $d_t$, respectively, each one being ranked by decreasing order.

We obtain a limit theorem concerning each of the laws of these three sequences, as $t \to \infty$. The result is expressed in terms of a positive, $\sigma$-finite measure $\Pi$ on the set $S^\downarrow$ of decreasing sequences. $\Pi$ is closely related with the Poisson–Dirichlet laws on $S^\downarrow$.

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Résumé

Asymptotiques pour la distribution des longueurs des excursions d’un processus de Bessel de dimension $d$ $(0 < d < 2)$. Soit $(R_t, t \geq 0)$ un processus de Bessel de dimension $d \in (0, 2)$. Pour tout $t \geq 0$, on considère les temps $g_t = \sup\{s \leq t: R_s = 0\}$ et $d_t = \inf\{s > t: R_s = 0\}$, ainsi que les trois suites : $(V^n_{g_t}, n \geq 1)$, $(V^n_{d_t}, n \geq 2)$, et $(V^n_{d_t}, n \geq 2)$ des longueurs d’excursions de $R$ hors de 0, avant $g_t$, resp. avant $t$, resp. avant $d_t$, rangées par ordre décroissant.

Nous obtenons un théorème limite concernant chacune des lois de ces trois suites, lorsque $t \to \infty$. Ce théorème s’exprime à l’aide d’une mesure positive, $\sigma$-finie, $\Pi$ sur $S^\downarrow = \{s = (s_1, s_2, \ldots, s_n, \ldots); s_1 \geq s_2 \geq \cdots \geq s_n \geq \cdots \geq 0\}$. $\Pi$ est intimement liée aux lois de Poisson–Dirichlet sur $S^\downarrow$. Pour citer cet article : B. Roynette et al., C. R. Acad. Sci. Paris, Ser. I 343 (2006). © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Version française abrégée

Soit $(R_t, t \geq 0)$; $P^{(x)}$ un processus de Bessel de dimension $d = 2(1 - \alpha)$ $(0 < \alpha < 1$, ou $0 < d < 2)$. Pour tout $t$, soit $V^1_t \geq V^2_t \geq V^3_t \geq \cdots$ la suite des longueurs des excursions hors de 0 du processus $R$ pendant l’intervalle de temps...
Théorème 0.1. (i) Asymptotique des lois des suites \((V_i^t; i = 1, 2, \ldots, V_n^t; \ldots)\), figure l’âge \((t - g_i)\) de l’excursion débutant en \(g_i = \sup\{s \leq t : R_s = 0\}\) et finissant en \(d_i = \inf\{s > t : R_s = 0\}\).

Nous nous intéressons, dans cette Note, à l’étude asymptotique, quand \(t \to \infty\), des lois des 3 suites \((V_i^t; i = 1, 2, \ldots, V_n^t; i = 2, 3, \ldots)\) et \((V_i^{d_i}; i = 2, 3, \ldots\) et \((V_i^{g_i}; i = 1, 2, \ldots\). Nos principaux résultats sont présentés dans les Théorèmes 0.1 et 0.2 ci-dessous.

Asymptotique de la loi de \((V_i^1, V_i^2, \ldots)\)

On note \(W_i(i = 1, 2, \ldots)\) la suite des coordonnées de l’ensemble \(S^i = \{s = (s_1, s_2, \ldots, s_n, \ldots) ; s_1 \geq s_2 \geq s_3 \geq \cdots\}\), c’est-à-dire : \(W_i(s) = s_i\).

Théorème 0.1. (i) Il existe une mesure positive, \(\sigma\)-finie, \(\Pi\) sur \(S^i\) ne dépendant pas de \(\alpha \in (0, 1)\), telle que, pour toute \(F: S^i \to \mathbb{R}_+\), borélienne, bornée, à support compact en la première variable, on ait :

\[
E_{\Pi}[F(W^1, W^2, \ldots, W^n)] := \lim_{t \to \infty} t^\alpha E^{(\alpha)}[F((V_i^t)^\alpha; i = 1, 2, \ldots)]
\]

\[
= \int_0^\infty E\left[F\left(x, x \frac{T_2}{T_3}, x \frac{T_2}{T_4}, \ldots, x \frac{T_2}{T_{n+1}}, \ldots\right)\right] dx
\]

\[
= \int_0^\infty E\left[F(x, x \rho_2, x \rho_2 \rho_3, \ldots, x \rho_2 \cdots \rho_{n}, \ldots\right] dx
\]

où, dans les deux dernières expressions :

- \(\rho_2, \rho_3, \ldots, \rho_n, \ldots\) sont des v.a. indépendantes, et pour tout \(n\), \(\rho_n \overset{(loi)}{=} U_{1/n}\) suit la loi beta de paramètre \((n, 1)\) et \(U\) est uniforme sur \((0, 1)\);

- \((T_i, i \geq 1)\) est la suite croissante des instants de sauts d’un processus de Poisson standard.

(ii) Sous la mesure \(\Pi\), la suite \((W^n)\) possède les propriétés suivantes :

- \(W^1\) est distribuée selon la mesure de Lebesgue sur \(\mathbb{R}_+\), et est indépendante de

\[
\begin{pmatrix}
W^2 & W^3 & W^n \\
W^1 & W^1 & W^n \\
\end{pmatrix}
\]

\((loi)\) \(
\begin{pmatrix}
T_2 & T_2 & T_2 \\
T_3 & T_4 & T_{n+1} \\
\end{pmatrix}
\)

- Plus généralement, pour tout \(n\), \(W^n\) est distribuée selon \(n\) fois la mesure de Lebesgue sur \(\mathbb{R}_+\), et est indépendante de

\[
\begin{pmatrix}
W^{n+1} & W^{n+k} \\
W^n & W^n \\
\end{pmatrix}
\]

\((loi)\) \(
\begin{pmatrix}
T_{n+1} & T_{n+1} & T_{n+1} \\
T_{n+2} & T_{n+3} & T_{n+k+1} \\
\end{pmatrix}
\)

Asymptotique des lois des suites \((V_i^t; i \geq 2)\) et \((V_i^{d_i}; i \geq 2)\)

Théorème 0.2. Soit \(F\) comme dans le Théorème 0.1. Les quantités \(t^\alpha E^{(\alpha)}[F(V_i^t; i \geq 2)]\) et \(t^\alpha E^{(\alpha)}[F(V_i^{d_i}; i \geq 2)]\) convergent toutes deux, lorsque \(t \to \infty\), vers la même limite, qui est égale à :

\[
E_{\Pi}[F(W^1, W^2, \ldots)]
\]

On notera que les suites \((V_i^t; i \geq 2)\) et \((V_i^{d_i}; i \geq 2)\) sont « décalées d’un indice », et ne font figurer ni \(V_i^1\), ni \(V_i^{d_i}\). L’explication intuitive de ce fait est donnée par le Théorème 6 ci-dessous, et résulte du Lemme 5.
1. The Poisson–Dirichlet distributions $P_{α,0}$ and $P_{α,α}$

1) Throughout this Note, we consider $(R_t,t \geq 0; P^{(α)})$ a Bessel process of dimension $d$, with $0 < d < 2$, $d = 2(1 - α)$, started from 0. Let $V^{1}_t \geq V^{2}_t \geq V^{3}_t \cdots$ denote the sequence of lengths of its excursions away from 0, during the time interval $[0,t]$, ranked in decreasing order. In particular, the so-called age $(t - g_t)$, where $g_t = \sup\{s \leq t: R_s = 0\}$, is included in the sequence $(V^1_t, V^2_t, \ldots)$. 

2) By scaling, the law of: 
\[
\frac{1}{t} (V^1_t, V^2_t, \ldots)
\]

does not depend on $t$, it has some remarkable properties, and is found naturally in a number of probabilistic studies. Its distribution, the Poisson–Dirichlet distribution with parameter $(α,0)$, is denoted by $P_{α,0}$ in [6]. 

3) Likewise, if in (1), we replace $t$ by $g_t := \sup\{s \leq t: R_s = 0\}$

then, the variable:
\[
\frac{1}{g_t} (V^1_{g_t}, V^2_{g_t}, \ldots)
\]

is independent from $g_t$, and is distributed as:
\[
(v^1, v^2, \ldots, v^n, \ldots)
\]

the sequence of ranked excursions of the standard BES(d) bridge; its distribution, the Poisson–Dirichlet distribution with parameter $(α,α)$ is denoted by $P_{α,α}$ in [6]. 

4) Here is a description of $P_{α,0}$ and $P_{α,α}$ on the canonical space $S^↓$ of decreasing sequences $s = \{s_1 \geq s_2 \geq \cdots \geq s_n \geq \cdots \geq 0\}$, where we denote $W^i(s) = s_i$ the sequence of coordinates.

Theorem 1. ([6]) Let $\{T_i = \varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_i, i = 1, 2, \ldots\}$ denote the sequence of jump times of a standard Poisson process with parameter 1, i.e.: the $\varepsilon_i$ are i.i.d. standard exponential variables. Then:

(i) Under $P_{α,0}$, the sequence $(W^i = (V^i)^α; i = 1, 2, \ldots)$ is distributed as
\[
\left(\left\{T_i \left(\sum_{m=1}^{∞} T_m^{-1/α}\right)^{-1/α}\right\}; i = 1, 2, \ldots\right).
\]

Consequently, $L := \lim_{n \to ∞} nW^n$ exists $P_{α,0}$ a.s., and is distributed (jointly with the $W^i$’s) as
\[ \left( \sum_{m=1}^{\infty} T_m^{-1/\alpha} \right)^{-\alpha}. \] In other terms:

\[
\left( L, \frac{W^i}{L}; i = 1, 2, \ldots \right) \stackrel{\text{law}}{=} \left( \left( \sum_{m=1}^{\infty} T_m^{-1/\alpha} \right)^{-\alpha}, \frac{1}{T_i}; i = 1, 2, \ldots \right). \tag{8}
\]

(ii) A closely related and useful description of \( P_{\alpha,0} \) is:

\[
(W^2, W^3, \ldots, W^{n+1}, \ldots) \stackrel{\text{law}}{=} \left( W^1 \frac{T_1}{T_2}, W^1 \frac{T_1}{T_3}, \ldots, W^1 \frac{T_1}{T_{n+1}}, \ldots \right) \stackrel{\text{law}}{=} \left( W^1 \rho_1, W^1 \rho_1 \rho_2, \ldots, W^1 \rho_1 \rho_2 \cdots \rho_n, \ldots \right) \tag{9}
\]

where \( (\rho_n = \frac{T_n}{T_{n+1}}, n \geq 1) \) is a sequence of independent variables, and \( \rho_n \) is beta\((n, 1)\) distributed. Note that, from (9), one has:

\[
(W^1)^{1/\alpha} + \sum_{n=1}^{\infty} (W^1)^{1/\alpha} (\rho_1 \cdots \rho_n)^{1/\alpha} = 1 \tag{10}
\]

so that \( W^1 \) is determined from the \( \rho_i \)'s.

(iii) \( P_{\alpha,\alpha} \) is absolutely continuous with respect to \( P_{\alpha,0} \), with:

\[
P_{\alpha,\alpha} = C_{\alpha} L \cdot P_{\alpha,0} \tag{11}
\]

where:

\[
C_{\alpha} = B(1 + \alpha, 1 - \alpha) = \Gamma(1 + \alpha) \Gamma(1 - \alpha) = \frac{\pi \alpha}{\sin(\pi \alpha)}. \tag{12}
\]

2. Asymptotic distribution of \( (V^1_{g_t}, V^2_{g_t}, \ldots) \)

1) Motivated by the study of the existence and description of the penalised laws:

\[
\lim_{t \to \infty} \frac{1_{(V^i_{g_t}) \leq x_i; i \leq k}^* P^{(\alpha)}}{P^{(\alpha)}(V^i_{g_t} \leq x_i; i \leq k)} \tag{13}
\]

where \( x_1 \geq x_2 \geq \cdots \geq x_k \) is a fixed, finite, decreasing sequence of positive reals (see [8,9]), we studied the asymptotics of the denominator of (13), and obtained the following:

**Theorem 2.** (i) There exists a positive, \( \sigma \)-finite measure \( \Pi \) on \( S^1 \), which does not depend on \( \alpha \in (0, 1) \), such that, for every \( F : S^1 \to \mathbb{R}_+ \), bounded, Borel, and with compact support in the first variable:

\[
E_{\Pi}[F(W^1, W^2, \ldots)] := \lim_{t \to \infty} t^\alpha E^{(\alpha)}[F((V^i_{g_t})^\alpha; i = 1, 2, \ldots)]
\]

\[
= \int_0^\infty E\left[F\left(x, x \frac{T_2}{T_3}, x \frac{T_2}{T_4}, \ldots, x \frac{T_2}{T_{n+1}}, \ldots\right)\right] dx
\]

\[
= \int_0^\infty dx E\left[F(x, x \rho_2, x \rho_2 \rho_3, \ldots, x \rho_2 \cdots \rho_n, \ldots)\right] \tag{14}
\]

with the same notation as in Theorem 1.

(ii) The measure \( \Pi \) enjoys the following properties:

(a) Under \( \Pi \), \( W^1 \) is distributed as Lebesgue measure on \( \mathbb{R}_+ \), and is independent from:

\[
\left( \begin{array}{cccc}
W^2 \\
\vdots \\
W^k \\
\end{array} \right) \stackrel{\text{law}}{=} \left( \begin{array}{cccc}
T_2 \\
\vdots \\
T_{k+1} \\
\end{array} \right) \tag{15}
\]
(b) More generally, under $\Pi$, for any $n$, $W^n$ is distributed as $n$ times Lebesgue’s measure on $\mathbb{R}_+$ and is independent of:

$$\left(\frac{W^{n+1}}{W^n}, \ldots, \frac{W^{n+k}}{W^n}\right) \overset{(law)}{=} \left(\frac{T_{n+1}}{T_{n+2}}, \ldots, \frac{T_{n+1}}{T_{n+k+1}}\right)$$

$$\overset{(law)}{=} (\rho_{n+1}, \rho_{n+1}\rho_{n+2}, \ldots, \rho_{n+1}\cdots\rho_{n+k}, \ldots). \tag{16}$$

(c) Under $\Pi$, the density of $(W^1, \ldots, W^n)$ is:

$$f_n(s_1, \ldots, s_n) = \frac{(n!)^n}{(s_1 s_2 \cdots s_{n-1})^n} \mathbf{1}_{0 \leq s_1 \leq \cdots \leq s_1}. \tag{17}$$

(d) Shifting the sequence $(W^k, k \geq 1)$ into $(W^{n+k}, k \geq 1)$, for any given $n \geq 1$, has the following effect on $\Pi$:

$$E_\Pi \left[ h(W^n, \ldots, W^{n+p}) \right] = \left( \frac{n + p}{p + 1} \right) E_\Pi \left[ \left( \frac{W^{p+1}}{W^1} \right)^{n-1} h(W^1, \ldots, W^{p+1}) \right] \tag{18}$$

for any $h: \mathbb{R}_+^{p+1} \to \mathbb{R}_+$, Borel.

We have obtained two proofs of Theorem 2, and we now present their main ingredients:

(i) Our first proof relies on the description of $P_{\alpha, \alpha}$ stated in the above Theorem 1, and upon the elementary remark: since $g_t/t$ is beta($\alpha, 1 - \alpha$) distributed, then:

$$t^\alpha E_\rho \left[ \varphi((g_t)^\alpha) \right] \to_{t \to \infty} \frac{1}{B(1 + \alpha, 1 - \alpha)} \int_0^\infty dx \varphi(x) \tag{19}$$

for any $\varphi: \mathbb{R}_+ \to \mathbb{R}_+$, Borel, with compact support.

(ii) Our second proof does not depend on Theorem 1; instead, it relies upon the following:

Lemma 3. Let $n \geq 1$, and $y_1 \geq y_2 \geq \cdots \geq y_n \geq 0$, a decreasing sequence of $n$ real numbers. Then (all functions introduced below do not depend on $\alpha$):

(i) There exist functions $P_n$ of $n$ variables such that:

$$P^{(\alpha)} \left[ (V_{g_t})^\alpha \right] \leq y_1, \ldots, (V_{g_t}^n)^\alpha \leq y_n \right) \sim \frac{1}{t^{(p+1)}} P_n(y_1, \ldots, y_n). \tag{20}$$

(ii) For any $p \geq 2$, there exists a polynomial $Q_p$ in $p$ variables, which is homogeneous of degree $(p - 2)$, such that:

$$P_n(y_1, \ldots, y_n) = y_n + \sum_{p=2}^n y_n^p \left( \frac{1}{y_n} - \frac{1}{y_{p-1}} \right) Q_p \left( \frac{1}{y_1}, \frac{1}{y_2}, \ldots, \frac{1}{y_{p-1}}, \frac{1}{y_n} \right). \tag{21}$$

In particular, $Q_2 \equiv 1$, $Q_3(x_1, x_2, x_3) = x_3 + x_2 - 2x_1$, $Q_4(x_1, x_2, x_3, x_4) = x_3^2 + x_3 x_4 + x_4^2 - 3x_3 x_1 - 3x_2 - 3x_1 x_4 + 6x_1 x_2$.

(iii) The polynomials $(Q_n, n \geq 2)$ satisfy the following recurrence relation:

$$Q_n(x_1, \ldots, x_n) = \sum_{p=1}^{n-2} \left( \begin{array}{c} n - 1 \\ p \\ \end{array} \right) (x_n - x_{n-1})^{p-1}(x_{n-1} - x_{n-p-1})$$

$$\cdots Q_{n-p}(x_1, x_2, \ldots, x_{n-p-2}, x_{n-p-1}, x_{n-1}) + (x_n - x_{n-1})^{n-2}. \tag{22}$$

(iv) The coefficient of $x_1 x_2 \cdots x_{n-2}$ in $Q_n(x_1, \ldots, x_n)$ is equal to $(-1)^n(n - 1)!$

(v) The $n$th mixed partial derivative of $P_n$, with respect to all variables is:

$$\frac{\partial^n P_n}{\partial y_1 \cdots \partial y_n}(y_1, \ldots, y_n) = \frac{n! y^{n-1}_n}{(y_1 y_2 \cdots y_{n-1})^2} \tag{23}$$

for $y_n < y_{n-1} < \cdots < y_1$. 
Our second proof of Theorem 2 easily follows from Lemma 3: in particular, with the help of (20) and (23), we obtain, that for any $F: S_0^1 \to \mathbb{R}_+$, bounded, and with compact support in the first variable:

$$t^\alpha E^{(\alpha)}[F((V^1_t)^\alpha, \ldots, (V^m_t)^\alpha)]$$

converges towards:

$$\int F(s_1, \ldots, s_n) \prod_{i=1}^{n} s_i^{\frac{\alpha-1}{2}} \prod_{i=1}^{n} dx_i \equiv \int F(s_1, \ldots, s_n) d\Pi(s).$$

3. Asymptotic distributions of $(V^2_t, V^3_t, \ldots)$ and $(V^2_{d_t}, V^3_{d_t}, \ldots)$

The next theorem is a companion to Theorem 2.

**Theorem 4.** With the same hypotheses concerning $F$ as in Theorem 2, both quantities:

$$\lim_{t \to \infty} t^\alpha E^{(\alpha)}[F((V^2_t)^\alpha, \ldots, (V^{n+1}_t)^\alpha, \ldots)]$$

and

$$\lim_{t \to \infty} t^\alpha E^{(\alpha)}[F((V^2_{d_t})^\alpha, \ldots, (V^{n+1}_{d_t})^\alpha, \ldots)]$$

exist and are equal to: $E[\Pi[F(W^1, W^2, \ldots, W^n, \ldots)]]$.

Note the shift of indices between the expressions found in (24) and (25) on one hand, and (26) on the other hand.

Intuitively, the explanation of the absence of $V^1_t$ in (24) comes from the fact that asymptotically, $V^1_t$ is much bigger than the other excursion lengths (see Theorem 6 below for a precise assertion of this fact; see [3] for the law of $V^1_t$), and that, with a large probability, $V^1_t = t - g_t$. Indeed, from [5]: $P(t - g_t = V^1_t) = E[\frac{V^1_t}{t}]$.

Hence, this sequence in $i$ is decreasing. More completely, we obtained the following asymptotics:

**Lemma 5.** For any $n \geq 2$, and $x_1 > x_2 > \cdots > x_n$:

(i) $P(\omega)(V^2_t \leq x_1, \ldots, V^{n+1}_t \leq x_n, t - g_t = V^1_t) = P(\omega)(V^1_{g_t} \leq x_1, \ldots, V^n_{g_t} \leq x_n, t - g_t > V^1_{g_t}) \sim \lim_{t \to \infty} P(\omega)(V^1_{g_t} \leq x_1, \ldots, V^n_{g_t} \leq x_n).$

(ii) $P(\omega)(V^2_t \leq x_1, \ldots, V^{n+1}_t \leq x_n, t - g_t \leq V^1_t) = o(t^{-\alpha}).$

Lemma 5 still holds as we replace in (27) and (28) the sequence $(V^2_t, \ldots, V^{n+1}_t)$ by $(V^2_{d_t}, \ldots, V^{n+1}_{d_t})$.

4. On one-dimensional asymptotics

1) Earlier in our study, we had obtained only one-dimensional asymptotics, that is estimates as $t \to \infty$, for:

$P(\omega)((V^1_t)^\alpha \leq x)$, and similarly with $V^n_t$, and $V^{n+1}_t$.

We now give some details about the different tools we used to obtain these estimates.

2) We denote by $\Phi(\alpha, \gamma; \cdot)$ $(\gamma \neq 0, -1, -2)$ the confluent hypergeometric function of index $(\alpha, \gamma)$:

$$\Phi(\alpha, \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k z^k}{(\gamma)_k k!} \quad (z \in \mathbb{C})$$

with $(\lambda)_k = \frac{\Gamma(\lambda+k)}{\Gamma(\lambda)}$ (cf. [4], p. 260).

Let $\lambda_0(\alpha)$ denote the first positive zero of:

$$\theta_\alpha(\lambda) = 1 - \lambda e^{-\lambda} \frac{1}{1-\alpha} \Phi(1-\alpha, 2-\alpha; \lambda).$$

We are now able to present our one-dimensional estimates for $n = 1$. 

$$t^\alpha E^{(\alpha)}[F((V^1_t)^\alpha)]$$

converges towards:

$$\int F(s_1) \prod_{i=1}^{n} s_i^{\frac{\alpha-1}{2}} \prod_{i=1}^{n} dx_i \equiv \int F(s_1) d\Pi(s).$$
Theorem 6. The three asymptotic results hold:

(i) \[ t^\alpha P^{(\alpha)}(V_{gt}^1 \leq x) \xrightarrow{t \to \infty} x^{\alpha}. \] (31)

(ii) There exists a constant \( C(\alpha) \) such that:

\[ P^{(\alpha)}(V^1_t \leq x) \sim C(\alpha) \exp\left(-\lambda_0(\alpha) \frac{x}{t}\right) \quad \text{and} \]

\[ P^{(\alpha)}(V_{d^1}^1 \leq x) \sim (C(\alpha) \exp(-\lambda_0(\alpha))) \exp\left(-\lambda_0(\alpha) \frac{x}{t}\right). \] (33)

Our main tool for the proof of Theorem 6 is Lemma 7.

Lemma 7. Let \( x \geq 0 \), and \( H^{(1)}_x = \inf\{t \geq 0 : t - g_t > x\} \). Then, there exist two functions \( \psi, \theta : \mathbb{R}_+ \to \mathbb{R}_+ \) such that:

(i) \[ \psi(x\beta) := E^{(\alpha)}(\exp(-\beta H^{(1)}_x)) = \frac{1}{\Phi(1, 1 - \alpha; \beta x)} \quad \text{and} \]

\[ \Phi(1, 1 - \alpha; \beta x) = 1 + (\beta x) e^{\beta x} \frac{1}{1 - \alpha} \Phi(1 - \alpha, 2 - \alpha; -\beta x). \] (34)

The function \( \beta \to \psi(\beta) \) is holomorphic in the strip: \( \text{Re} \beta > -\lambda_0(\alpha) \). In particular:

\[ \psi(x\beta) = 1 - \beta x E^{(\alpha)}(H^{(1)}_1) + o(\beta) = 1 - \frac{\beta x}{1 - \alpha} + o(\beta) \quad (\beta \to 0) \] (35)

(ii) \( \theta(x\beta) := E^{(\alpha)}\left[E^{(\alpha)}(\exp(-\beta T_0)) \right] = \Phi(1, 1 - \alpha; \beta x) - \Gamma(1 - \alpha)(\beta x)^\alpha \exp(\beta x) \] (36)

\[ = \Gamma(1 - \alpha)(\beta x)^\alpha + o(\beta^\alpha) \quad (\beta \to 0) \] (37)

where \( T_0 := \inf\{t : R_t = 0\} \) and \( P^{(\alpha)}_r \) denotes the law of the BES(\( \alpha \)) process starting from \( r \) at time \( t = 0 \).

Note that formula (36) had already been obtained under a slightly different form in \([6], \text{Proposition 11}\). We now prove point (i) of Theorem 6:

Let \( S_\beta \) be an exponential variable with parameter \( \beta \), independent from \( (R_t, t \geq 0) \). We have:

\[ P^{(\alpha)}(V_{S_\beta}^1 \leq x) = P^{(\alpha)}(S_\beta \leq d_{H^{(1)}_1}) = 1 - E^{(\alpha)}(\exp(-\beta d_{H^{(1)}_1})) \]

\[ = 1 - E^{(\alpha)}(\exp(-\beta H^{(1)}_1 + T_0 \circ \Theta_{H^{(1)}_1})) \]

\[ = 1 - E^{(\alpha)}(\exp(-\beta H^{(1)}_1)) E^{(\alpha)}(E^{(\alpha)}_{R^{(1)}_{H_{1}^{(1)}}}(e^{-\beta T_0})) \]

(from the independence of \( H^{(1)}_x \) and \( R_{H^{(1)}_1} \) (see, e.g. \([2]\))

\[ = 1 - \psi(x\beta) \theta(x\beta). \]

On the other hand,

\[ P^{(\alpha)}(V_{S_\beta}^1 \leq x) = \beta \int_0^\infty dt e^{-\beta t} P^{(\alpha)}(V_{S_\beta}^1 \leq x). \]

Hence:

\[ \int_0^\infty e^{-\beta t} P^{(\alpha)}(V^1_{S_\beta} \leq x) dt \sim \Gamma(1 - \alpha) x^\alpha \beta^{\alpha - 1} \]

and we conclude with the help of the Tauberian theorem.

The proofs of points (ii) and (iii) of Theorem 6 follow in the same manner, although we then use the Mellin–Fourier transform instead of the Tauberian theorem (and (34)).

3) Here are now our one-dimensional results for \( n \geq 2 \).
Theorem 8. The three asymptotic results hold:

(i) For any \( n \geq 1 \), \( \lim_{t \to \infty} t^\alpha P^{(\alpha)}(V_n^{\alpha} \leq x) = nx^\alpha. \) \hspace{1cm} (38)

(ii) For any \( n \geq 2 \), \( \lim_{t \to \infty} t^\alpha P^{(\alpha)}(V_t^{\alpha} \leq x) = (n - 1)x^\alpha. \) \hspace{1cm} (39)

(iii) For any \( n \geq 2 \), \( \lim_{t \to \infty} t^\alpha P^{(\alpha)}(V_n \leq x) = (n - 1)x^\alpha. \) \hspace{1cm} (40)

The proof of Theorem 8 is quite similar to that of Theorem 6. It hinges upon the following:

Lemma 9. Define, by iteration, for any \( n \geq 1 \), the sequence of stopping times:

\[
H(n+1) = dH(n) + H(1) \circ \Theta dH(n)
\]

where \( \Theta_u, u \geq 0 \) are the usual time-shift operators. Then:

(i) \( \psi_n(\beta x) := E^{(\alpha)}(\exp(-\beta H(n))) = \psi(\beta x)^n(\theta(\beta x))^{n-1}. \)

(ii) \( E^{(\alpha)}(\exp(-\beta dH(n))) = \psi(\beta x)^n(\theta(\beta x))^n. \)

(iii) \( E^{(\alpha)}(\exp(-\beta gH(n))) = \exp(\beta x) \psi_n(\beta x). \)

4) Our aim in making the asymptotic study of \( P^{(\alpha)}(V_1 \leq x_1, \ldots, V_n \leq x_n) \) was to penalise the Bessel process by the functional:

\[
\Gamma^{(n)}_t = 1_{V_1 \leq x_1, \ldots, V_n \leq x_n}.
\]

More precisely, we wish to show that, for any \( \Lambda_s \in F_s = \sigma\{R_u, u \leq s\} \), and for any \( s \geq 0 \):

\[
Q(\Lambda_s) := \lim_{t \to \infty} \frac{E^{(\alpha)}(1_{\Lambda_s} \Gamma^{(n)}_t)}{E^{(\alpha)}(\Gamma^{(n)}_t)} = E^{(\alpha)}(1_{\Lambda_s} M^{(x_1, \ldots, x_n)}_s)
\]

where \( (M^{(x_1, \ldots, x_n)}_s, s \geq 0) \) is a \( P^{(\alpha)} \)-martingale, and then to describe the canonical process \( (R_t, t \geq 0) \) under the probability \( Q \) induced by (42). We have been able to go through this study for \( \alpha = 1/2 \), and \( n = 1 \) in [8], that is, when \( (R_t, t \geq 0) \) is reflecting Brownian motion, as well as in [9] for any \( \alpha \in (0, 1) \), and \( n = 1 \). We are presently investigating the general case \( (\alpha \in (0, 1), n \geq 1) \).

References


[9] B. Roynette, P. Vallois, M. Yor, Penalisation of a Bessel process of dimension \( d = 2(1 - \alpha) \) \((0 < d < 2)\) by a function of its longest excursion, IX (March 2006), in preparation.