

Probability Theory

On the comparison theorem for multidimensional BSDEs

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Abstract

In this Note, we give a necessary and sufficient condition under which the comparison theorem holds for multidimensional backward stochastic differential equations (BSDEs) and for matrix-valued BSDEs. *To cite this article: Y. Hu, S. Peng, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Résumé

Sur le théorème de comparaison pour les équations différentielles stochastiques rétrogrades. Dans cette Note, nous donnons une condition nécessaire et suffisante sous laquelle le théorème de comparaison fonctionne pour les équations différentielles stochastiques rétrogrades (EDSR) multidimensionnelles et pour les EDSR à valeurs matricielles. *Pour citer cet article: Y. Hu, S. Peng, C. R. Acad. Sci. Paris, Ser. I 343 (2006).*

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Le théorème de comparaison pour les EDSR à valeurs réelles est un classique de cette théorie. Il a été énoncé pour la première fois par Peng dans [6], et plusieurs fois généralisé depuis, pour prendre sa forme définitive dans [4]. Ce théorème permet de comparer des solutions d'EDSR dès qu'on peut comparer leurs générateurs. Le théorème de comparaison inverse pour les EDSR à valeurs réelles a aussi été étudié récemment, voir [1,3].

Le but de cette Note est de donner une condition nécessaire et suffisante sous laquelle le théorème de comparaison fonctionne pour les EDSR multidimensionnelles et pour les EDSR à valeurs matricielles.

Les hypothèses (A1)–(A3) sont données à la Section 2.

Théorème 1. *On se donne f^1 et f^2 vérifiant (A1)–(A3). Alors les deux conditions suivantes sont équivalentes :*

- (i) $\forall \tau \in [0, T], \forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$ t.q. $\xi^1 \geq \xi^2$, les solutions uniques (Y^1, Z^1) et (Y^2, Z^2) dans $L^2_{\text{ad}}(\Omega, C([0, \tau], \mathbb{R}^n)) \times L^2_{\text{ad}}(\Omega \times (0, \tau), \mathbb{R}^{n \times d})$ de l'EDSR (4) associée aux valeurs terminales ξ^1 et ξ^2 sur l'intervalle $[0, \tau]$ vérifient : $Y_t^1 \geq Y_t^2, t \in [0, \tau]$;

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(ii) $\forall t, \forall(y, y'), \forall(z, z')$,

$$-4\langle y^-, f^1(t, y^+ + y', z) - f^2(t, y', z') \rangle \leq 2 \sum_{k=1}^n 1_{y_k < 0} |z_k - z'_k|^2 + C|y^-|^2, \quad \mathbb{P}\text{-a.s.}$$

où $C > 0$ est une constante.

Dans le cas où $f^1 = f^2$, on a :

Théorème 2. *On se donne f vérifiant (A1)–(A3). Alors les deux conditions suivantes sont équivalentes :*

- (i) $\forall \tau \in [0, T], \forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$ t.q. $\xi^1 \geq \xi^2$, les solutions uniques (Y^1, Z^1) et (Y^2, Z^2) dans $L^2_{\text{ad}}(\Omega, C([0, \tau], \mathbb{R}^n)) \times L^2_{\text{ad}}(\Omega \times (0, \tau), \mathbb{R}^{n \times d})$ de l'EDSR (4) associée aux valeurs terminales ξ^1 et ξ^2 sur l'intervalle $[0, \tau]$ vérifient : $Y^1_t \geq Y^2_t, t \in [0, \tau]$;
- (ii) pour tout $k = 1, 2, \dots, n$,

f_k ne dépend que de z_k et

$$f_k(t, \delta^k y + y', z_k) \geq f_k(t, y', z_k), \quad \forall \delta^k y \in \mathbb{R}^n \text{ t.q. } \delta^k y \geq 0, (\delta^k y)_k = 0.$$

1. Introduction

The comparison theorem for real-valued BSDEs turns out to be one of the classic results of this theory. It is due to S. Peng [6] and then generalized by Pardoux–Peng [5] and El Karoui–Peng–Quenez [4]. It allows to compare the solutions of two real-valued BSDEs whenever we can compare the terminal conditions and the generators. The converse comparison theorem for real-valued BSDEs has been also studied recently, see, e.g., [1,3].

The objective of this Note is to give a necessary and sufficient condition under which the comparison theorem holds for multi-dimensional BSDEs.

For this, we apply the Backward Stochastic Viability Property (BSVP) studied in [2] together with a coupling method.

The Note is organized as follows: in the next section, we study the comparison theorem for multi-dimensional BSDEs; in Section 3, we study the comparison theorem for symmetric matrix valued BSDEs; and in Appendix A, we calculate the first and second derivatives of some functions needed in Section 3.

2. Multidimensional BSDEs

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space carrying a standard d -dimensional Brownian motion, $(W_t)_{t \geq 0}$, starting from $W_0 = 0$, and let $(\mathcal{F}_t)_{t \geq 0}$ be the σ -algebra generated by $(W_t)_{t \geq 0}$. We do the usual \mathbb{P} -augmentation to each \mathcal{F}_t so that $(\mathcal{F}_t)_{t \geq 0}$ is right continuous and complete. By $T > 0$ we denote the finite real time horizon. Throughout this Note, for any Euclidean space H , we denote by $L^2_{\text{ad}}(\Omega, C([0, T], H))$ the closed linear subspace of adapted processes of $L^2(\Omega, \mathcal{F}, \mathbb{P}, C([0, T], H))$, and $L^2_{\text{ad}}(\Omega \times (0, T), H)$ is the Hilbert space of adapted measurable stochastic processes ψ such that $\|\psi\|_2 := (\mathbb{E} \int_0^T \|\psi(t)\|^2 dt)^{1/2} < \infty$.

Let us consider a function f , which will be in the sequel the generator of the BSDE, defined on $\Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^{n \times d}$, with values in \mathbb{R}^n , such that the process $(f(t, y, z))_{t \in [0, T]}$ is a progressively measurable process for each $(y, z) \in \mathbb{R}^n \times \mathbb{R}^{n \times d}$.

For the function f , we will use, throughout this Note, the following assumptions:

(A1) \mathbb{P} -a.s., $\forall(y, z), t \rightarrow f(t, y, z)$ is continuous.

(A2) There exists a constant $K \geq 0$, such that, \mathbb{P} -a.s., we have: $\forall t, \forall(y, y'), \forall(z, z'), |f(t, y, z) - f(t, y', z')| \leq K(|y - y'| + |z - z'|)$.

(A3) $\sup_{t \leq T} |f(t, 0, 0)| \in L^2(\Omega, \mathcal{F}, \mathbb{P})$.

We consider the following BSDEs:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \tag{1}$$

where ξ is a given \mathbb{R}^n -valued random variable. We use the following assumption for ξ .

(A4) $|\xi| \in L^2(\Omega, \mathcal{F}, P)$.

The objective of this section is to study when the comparison theorem holds for two BSDEs of type (1). For this, let us first recall a result concerning the backward stochastic viability property (BSVP) for the BSDE (1) in K due to Buckdahn et al. [2], where K is a nonempty closed convex set of \mathbb{R}^n .

2.1. Viability property in K

We recall here the definition of BSVP from [2]. Let K be a nonempty closed convex set of \mathbb{R}^n . We define $\Pi_K(a)$ as the projection of a point a onto K , and $d_K(\cdot)$ as the distance function of K .

Definition 2.1. The BSDE (1) enjoys the BSVP in K if and only if: for any $\tau \in [0, T]$, $\forall \xi \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; K)$, the unique solution $(Y, Z) \in L^2_{ad}(\Omega, C([0, \tau], \mathbb{R}^n)) \times L^2_{ad}(\Omega \times (0, \tau), \mathbb{R}^{n \times d})$ to the BSDE (1) over time interval $[0, \tau]$:

$$Y_t = \xi + \int_t^\tau f(s, Y_s, Z_s) ds - \int_t^\tau Z_s dW_s, \tag{2}$$

satisfies $Y_t \in K, \forall t \in [0, \tau]$, \mathbb{P} -a.s.

For completeness, we recall the following from [2]:

Proposition 2.1. Suppose that f satisfies (A1)–(A3). Then the BSDE (1) enjoys the BSVP in K if and only if $\forall (t, z) \in [0, T] \times \mathbb{R}^{n \times d}$ and $\forall y \in \mathbb{R}^n$ such that $d_K^2(\cdot)$ is twice differentiable at y ,

$$4\langle y - \Pi_K(y), f(t, \Pi_K(y), z) \rangle \leq \langle D^2 d_K^2(y)z, z \rangle + C d_K^2(y), \quad a.s., \tag{3}$$

where $C > 0$ is a constant which does not depend on (t, y, z) .

2.2. Comparison theorem in \mathbb{R}^n

Consider the following two BSDEs: $i = 1, 2$,

$$Y_t^i = \xi^i + \int_t^T f^i(s, Y_s^i, Z_s^i) ds - \int_t^T Z_s^i dW_s, \tag{4}$$

where f^1, f^2 satisfy (A1)–(A3), and ξ^1, ξ^2 satisfy (A4). In this subsection, we study the following problem: under which condition the comparison theorem holds? Interestingly, this problem is transformed to a viability problem in $\mathbb{R}_+^n \times \mathbb{R}^n$ of $(Y^1 - Y^2, Y^2)$.

Theorem 2.1. Suppose that f^1 and f^2 satisfy (A1)–(A3). Then the following are equivalent:

- (i) for any $\tau \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1) and (Y^2, Z^2) in $L^2_{ad}(\Omega, C([0, \tau], \mathbb{R}^n)) \times L^2_{ad}(\Omega \times (0, \tau), \mathbb{R}^{n \times d})$ to the BSDE (4) over time interval $[0, \tau]$ satisfy:

$$Y_t^1 \geq Y_t^2, \quad t \in [0, \tau];$$

(ii) $\forall t, \forall (y, y'), \forall (z, z')$,

$$-4\langle y^-, f^1(t, y^+ + y', z) - f^2(t, y', z') \rangle \leq 2 \sum_{k=1}^n 1_{y_k < 0} |z_k - z'_k|^2 + C|y^-|^2, \quad \mathbb{P}\text{-a.s.} \tag{5}$$

where $C > 0$ is a constant.

Proof. Set $\tilde{Y}_t = (Y_t^1 - Y_t^2, Y_t^2)$, $\tilde{Z}_t = (Z_t^1 - Z_t^2, Z_t^2)$.

Then (i) is equivalent to the following:

(iii) for any $\tau \in [0, T]$, $\forall \tilde{\xi} = (\tilde{\xi}^1, \tilde{\xi}^2)$ such that $\tilde{\xi}^1 \geq 0$, the unique solution (\tilde{Y}, \tilde{Z}) to the following BSDE over time interval $[0, \tau]$:

$$\tilde{Y}_t = \tilde{\xi} + \int_t^\tau \tilde{f}^i(s, \tilde{Y}_s, \tilde{Z}_s) ds - \int_t^\tau \tilde{Z}_s dW_s, \tag{6}$$

satisfies $\tilde{Y}^1 \geq 0$, where for $\tilde{y} = (\tilde{y}^1, \tilde{y}^2)$, $\tilde{z} = (\tilde{z}^1, \tilde{z}^2)$,

$$\tilde{f}(t, \tilde{y}, \tilde{z}) = (f^1(t, \tilde{y}^1 + \tilde{y}^2, \tilde{z}^1 + \tilde{z}^2) - f^2(t, \tilde{y}^2, \tilde{z}^2), f^2(t, \tilde{y}^2, \tilde{z}^2)).$$

Applying Proposition 2.1 to BSDE (6) and the convex closed set $\mathbb{R}_+^n \times \mathbb{R}^n$, it is clear that (iii) is equivalent to (ii). \square

Remark 2.1. Let $n = 1$, then (5) is equivalent to

$$f^1(t, y, z) \geq f^2(t, y, z). \tag{7}$$

This is established already in [1].

Now consider the special case when $f^1 = f^2$. Then we have:

Theorem 2.2. Suppose that f satisfies (A1)–(A3). Then the following are equivalent:

(i) for any $\tau \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{R}^n)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1) and (Y^2, Z^2) in $L_{\text{ad}}^2(\Omega, C([0, \tau], \mathbb{R}^n)) \times L_{\text{ad}}^2(\Omega \times (0, \tau), \mathbb{R}^{n \times d})$ to the BSDE (1) with terminal values ξ^1 and ξ^2 over time interval $[0, \tau]$ satisfy:

$$Y_t^1 \geq Y_t^2, \quad t \in [0, \tau];$$

(ii) for any $k = 1, 2, \dots, n$,

for any t and y , f_k depends only on z_k and

$$f_k(t, \delta^k y + y', z_k) \geq f_k(t, y', z_k), \quad \text{for any } \delta^k y \in \mathbb{R}^n \text{ such that } \delta^k y \geq 0, (\delta^k y)_k = 0. \tag{8}$$

Proof. From Theorem 2.1, (i) is equivalent to

$$-4\langle y^-, f(t, y^+ + y', z) - f(t, y', z') \rangle \leq 2 \sum_{k=1}^n 1_{y_k < 0} |z_k - z'_k|^2 + C|y^-|^2. \tag{9}$$

Now suppose that (9) holds. Let us pick $y_k < 0$, and $y = y_k e_k$, we get then:

$$4y_k (f_k(t, y', z) - f_k(t, y', z')) \leq 2|z_k - z'_k|^2 + C|y_k|^2. \tag{10}$$

We deduce then easily that f_k depends only on z_k . Moreover, for $\delta^k y \in \mathbb{R}^n$ such that $\delta^k y \geq 0$, $(\delta^k y)_k = 0$, putting in (9) $y = \delta^k y - \varepsilon e_k$, $\varepsilon > 0$, $z' = z$, dividing by $-\varepsilon$ and letting $\varepsilon \rightarrow 0^+$, we deduce also that: $f_k(t, \delta^k y + y', z_k) \geq f_k(t, y', z_k)$. Hence we proved (ii).

Finally it is easy to check that if (ii) holds, then (9) holds. \square

3. Matrix-valued BSDEs

In order to discuss the case of matrix-valued BSDEs, we should calculate the first and second derivatives of the function $d_{\mathbb{S}_+^n}^2(y)$, $y \in \mathbb{S}^n$, where \mathbb{S}^n is the space of symmetric real $n \times n$ matrices, and \mathbb{S}_+^n is the subset of \mathbb{S}^n containing the nonnegative elements in \mathbb{S}^n . This is done in the Appendix A. Without loss of generality, we set $d = 1$ in this section.

Let us again consider a function F , which will be in the sequel the generator of the BSDE, defined on $\Omega \times [0, T] \times \mathbb{S}^n \times \mathbb{S}^n$, with values in \mathbb{S}^n , such that the process $(F(t, y, z))_{t \in [0, T]}$ is a progressively measurable process for each $(y, z) \in \mathbb{S}^n \times \mathbb{S}^n$.

We consider the following matrix-valued BSDE:

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s, \tag{11}$$

where ξ is a given \mathbb{S}^n -valued random variable. We use the assumption (A4) for ξ .

The objective of this section is to study when the comparison theorem holds for two BSDEs of type (11).

Consider the following two BSDEs: $i = 1, 2$,

$$Y_t^i = \xi^i + \int_t^T F^i(s, Y_s^i, Z_s^i) ds - \int_t^T Z_s^i dW_s, \tag{12}$$

where F^1, F^2 satisfy (A1)–(A3), and ξ^1, ξ^2 satisfy (A4). In this section, we study the same problem as in the preceding section: under which condition does the comparison theorem hold for matrix-valued BSDEs? Interestingly, this problem is transformed again to a viability problem in $\mathbb{S}_+^n \times \mathbb{S}^n$.

Theorem 3.1. *Suppose that F^1 and F^2 satisfy (A1)–(A3). Then the following are equivalent:*

- (i) *for any $\tau \in [0, T]$, $\forall \xi^1, \xi^2 \in L^2(\Omega, \mathcal{F}_\tau, \mathbb{P}; \mathbb{S}^n)$ such that $\xi^1 \geq \xi^2$, the unique solutions (Y^1, Z^1) and (Y^2, Z^2) in $L_{\text{ad}}^2(\Omega, C([0, \tau], \mathbb{S}^n)) \times L_{\text{ad}}^2(\Omega \times (0, \tau), \mathbb{S}^n)$ to the BSDE (12) over time interval $[0, \tau]$ satisfy:*

$$Y_t^1 \geq Y_t^2, \quad t \in [0, \tau];$$
- (ii) $-4\langle y^-, F^1(t, y^+ + y', z) - F^2(t, y', z') \rangle \leq \langle D^2 d_{\mathbb{S}_+^n}^2(y)(z - z'), z - z' \rangle + C \|y^-\|^2.$ (13)

Remark 3.1. Let us suppose furthermore that F^1 and F^2 are independent of z . Then (ii) is reduced to:

$$(ii') \quad -4\langle y^-, F^1(t, y^+ + y') - F^2(t, y') \rangle \leq C \|y^-\|^2.$$

Appendix A

For $y, \bar{y} \in \mathbb{S}^n$, we denote $\|y\| = (\text{tr}(y^2))^{1/2}$, $\langle y, \bar{y} \rangle = \text{tr}(y\bar{y})$.

For any $y \in \mathbb{S}^n$, y has an expression: $y(\lambda, A) = e^A \sum_{i=1}^n \lambda_i e_i e_i^T e^{-A}$, where A is an antisymmetric real $n \times n$ matrix ($A^T = -A$), $\lambda_i \in \mathbb{R}$, $\{e_1, e_2, \dots, e_n\}$ is the standard basis of \mathbb{R}^n .

We define $y^+(\lambda, A) = e^A \sum_{i=1}^n \lambda_i^+ e_i e_i^T e^{-A}$, $y^-(\lambda, A) = e^A \sum_{i=1}^n \lambda_i^- e_i e_i^T e^{-A}$.

If $y \in \mathbb{S}_+^n$, then $\lambda_i \geq 0$. We define $y^{1/2}(\lambda, A) = e^A \sum_{i=1}^n (\lambda_i)^{1/2} e_i e_i^T e^{-A}$.

Let us first give an expression for $d_{\mathbb{S}_+^n}^2(y)$, $y \in \mathbb{S}^n$.

Lemma A.1. *For any $y \in \mathbb{S}^n$, $d_{\mathbb{S}_+^n}^2(y) = \|y^-\|^2$, and $\Pi_{\mathbb{S}_+^n}(y) = y^+$.*

The proof of this lemma is quite elementary and we omit it here.

Let us now calculate the first derivatives of $d_{\mathbb{S}_+^n}^2(y)$. For this, we fix an element $y(\lambda, A) = e^A \sum_{k=1}^n \lambda_k e_k e_k^T e^{-A}$, where $\lambda_k \neq 0$. We take a local orthonormal basis as follows: $E_{ii} = e^A e_i e_i^T e^{-A}$, $i = 1, \dots, n$, and $E_{ij} = \frac{1}{\sqrt{2}} e^A (e_i e_j^T + e_j e_i^T) e^{-A}$, $1 \leq i < j \leq n$.

On the one hand, we have: $\frac{d}{d\varepsilon} d_{\mathbb{S}_+^n}^2 (y(\lambda + \varepsilon e_i, A))|_{\varepsilon=0} = -2\lambda_i^-$, which gives us that

$$\left. \frac{d}{d\varepsilon} d_{\mathbb{S}_+^n}^2 (y(\lambda, A) + \varepsilon E_{ii}) \right|_{\varepsilon=0} = -2\lambda_i^-. \quad (14)$$

On the other hand, we have: for any antisymmetric real $n \times n$ matrix \bar{A} ,

$$\left. \frac{d}{d\varepsilon} d_{\mathbb{S}_+^n}^2 (e^A y(\lambda, \varepsilon \bar{A}) e^{-A}) \right|_{\varepsilon=0} = 0. \quad (15)$$

But

$$\left. \frac{d}{d\varepsilon} e^A y(\lambda, \varepsilon \bar{A}) e^{-A} \right|_{\varepsilon=0} = e^A \left(\bar{A} \sum_{k=1}^n \lambda_k e_k e_k^T - \sum_{k=1}^n \lambda_k e_k e_k^T \bar{A} \right) e^{-A},$$

and elementary algebra shows that for $\bar{A} = A_{ij} = e_i e_j^T - e_j e_i^T$,

$$A_{ij} \sum_{k=1}^n \lambda_k e_k e_k^T - \sum_{k=1}^n \lambda_k e_k e_k^T A_{ij} = (\lambda_j - \lambda_i)(e_i e_j^T + e_j e_i^T).$$

Taking into consideration the definition of E_{ij} , we have: $\frac{d}{d\varepsilon} e^A y(\lambda, \varepsilon \bar{A}) e^{-A}|_{\varepsilon=0} = (\lambda_j - \lambda_i)\sqrt{2} E_{ij}$.

Coming back to (15), we get then:

$$\left. \frac{d}{d\varepsilon} d_{\mathbb{S}_+^n}^2 (y(\lambda, A) + \varepsilon(\lambda_j - \lambda_i)\sqrt{2} E_{ij}) \right|_{\varepsilon=0} = 0.$$

Summarizing the above calculus, we have:

Lemma A.2. For $y(\lambda, A) = e^A \sum_{k=1}^n \lambda_k e_k e_k^T e^{-A}$, where $\lambda_1 < \lambda_2 < \dots < \lambda_n$, and $\lambda_k \neq 0$, $k = 1, 2, \dots, n$, $d_{\mathbb{S}_+^n}^2$ is differentiable at $y(\lambda, A)$, and $\nabla d_{\mathbb{S}_+^n}^2 (y) = -2y^-$.

The same type of calculus leads us to the expression of the second derivatives of $d_{\mathbb{S}_+^n}^2$.

Lemma A.3. For $y(\lambda, A) = e^A \sum_{k=1}^n \lambda_k e_k e_k^T e^{-A}$, where $\lambda_1 < \lambda_2 < \dots < \lambda_n$, and $\lambda_k \neq 0$, $k = 1, 2, \dots, n$, $d_{\mathbb{S}_+^n}^2$ is twice differentiable at $y(\lambda, A)$, and

$$D^2 d_{\mathbb{S}_+^n}^2 (y) E_{ii} = 21_{\lambda_i < 0} E_{ii}, \quad D^2 d_{\mathbb{S}_+^n}^2 (y) E_{ij} = -2 \frac{\lambda_j^- - \lambda_i^-}{\lambda_j - \lambda_i} E_{ij}, \quad i < j.$$

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