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Numerical Analysis

Existence of a solution 'in the large' for the 3D large-scale ocean dynamics equations

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Abstract

For the 3D system of equations describing large-scale ocean dynamics in the Cartesian coordinate system existence and uniqueness of a solution on an arbitrary time interval $[0,T]$ is proved and the norm $\|\hat{\mathbf{u}}_x\|$ is shown to be continuous in time on $[0,T]$. ***To cite this article: G.M. Kobelkov, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

L'existence d'une solution en 3D pour la dynamique de l'océan à grande échelle. L'auteur considère le système 3D d'équations décrivant la dynamique de l'océan à grande échelle en coordonnées cartésiennes. Il démontre, pour tout coefficient de viscosité et toute donnée initiale, l'existence et l'unicité d'une solution sur un intervalle de temps $[0,T]$ arbitrairement, ainsi que la continuité en temps sur l'intervalle $[0,T]$ de la norme $\|\hat{\mathbf{u}}_x\|$. ***Pour citer cet article : G.M. Kobelkov, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

In a number of papers of J.-L. Lions, R. Temam, etc. [5,6,4,3,2] study of the system of large-scale ocean dynamics was carried out. Theorems on existence and uniqueness of a solution 'in the small' and 'in the large' under the assumptions on smallness of a domain, time interval or initial data and even under the assumption of smallness of a domain in z direction were proved. In this Note we prove the existence and uniqueness of a solution for the 3D primitive equations 'in the large' without any smallness assumption.

Only for simplicity we consider the system of large-scale ocean dynamics in the Cartesian coordinates and assume the right-hand side to be equal to zero. We also simplify the system by reducing the heat equation and the salinity one to the density equation being the sum of them with some coefficients. In what follows, all considerations are also valid for the original system.

Let Ω be a cylinder in $\mathbb{R}^3$ of the form $\Omega = \Omega' \times [0,1]$, where Ω' is a domain on the x, y-plane with a piecewise smooth boundary. The boundary $\partial\Omega$ is represented as $\partial\Omega = S_1 \cup S$, $S = \partial\Omega' \times [0,1]$, so S_1 consists of the upper

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and bottom surfaces of the cylinder and S is its lateral surface, Q_t denotes the domain $Q_t = \Omega \times [0,t]$; so, $Q_T = \Omega \times [0,T]$.

Let $\mathbf{u} = (u_1, u_2, u_3)$ be a velocity vector; by $\hat{\mathbf{u}}$ we denote the vector–function $\hat{\mathbf{u}} = (u_1, u_2)$. In what follows, one assumes that the indices i and j range $1, 2$, and the index k—from 1 to 3. Space variables are denoted by $x = (x_1, x_2, x_3)$ as well as x, y, z and $x' = (x_1, x_2)$.

We use the following notations:

$$\partial_k v = \frac{\partial v}{\partial x_k}, \quad v_x = \nabla v, \quad v_{x'} = \nabla' v = (\partial_1 v, \partial_2 v), \quad \operatorname{div}' \hat{\mathbf{u}} = \partial_1 u_1 + \partial_2 u_2,$$
$$\Delta' = \partial_1^2 + \partial_2^2, \quad \|v\|_q = \|v\|_{L_q(\Omega)}, \quad \|v\| = \|v\|_{L_2}.$$

As usual, by c with and without indices we denote various constants in the inequalities not depending on the functions entering these inequalities.

The system of equations describing the large-scale ocean dynamics has the form [7,5]

$$\begin{aligned} &\hat{\mathbf{u}}_t - \nu\Delta\hat{\mathbf{u}} + l\hat{\mathbf{u}} + \nabla' p + u_k \hat{\mathbf{u}}_{x_k} = \mathbf{0}, \qquad \frac{\partial p}{\partial x_3} = -g\rho, \\ &\operatorname{div}\mathbf{u} = 0, \qquad \rho_t - \operatorname{div}(\nu_1 \nabla\rho) + u_k \rho_{x_k} = 0. \end{aligned} \tag{1'}$$

Boundary and initial conditions for $(1')$ are

$$\begin{aligned} &\hat{\mathbf{u}} \cdot \mathbf{n} = \frac{\partial \hat{\mathbf{u}}}{\partial n} \times \mathbf{n} = 0 \quad \text{on } S, \qquad \frac{\partial \hat{\mathbf{u}}}{\partial n} = \mathbf{0} \quad \text{on } S_1, \qquad u_3 = 0 \quad \text{on } S_1, \qquad \frac{\partial \rho}{\partial n} = 0 \quad \text{on } \partial\Omega, \\ &\hat{\mathbf{u}}(0,\, x) = \hat{\mathbf{u}}_0(x), \quad \int_0^1 \operatorname{div}' \hat{\mathbf{u}}_0 \, \mathrm{d}z = 0, \quad \rho(x,0) = \rho_0(x); \end{aligned} \tag{1''}$$

we assume summation over repeating indices in products; g is the acceleration of gravity, $\mathbf{n}$ is the outer unit normal to S, $\frac{\partial}{\partial n}$ is the normal derivative and $\mathbf{a} \times \mathbf{b} = a_1 b_2 - a_2 b_1$. The operator $l\hat{\mathbf{u}}$ is of the form $l\hat{\mathbf{u}} = \omega(u_2, -u_1)$ and is skew-symmetric.

Proof of the existence of a solution to $(1')$, $(1'')$ 'in the large' is based on the following chain of a priori estimates. The first is obtained by taking the scalar product in $L_2(\Omega)$ of the last equation $(1')$ and ρ^3:

$$\max_{0 \leqslant t \leqslant T} \|\rho(t)\|_4^4 + \nu_1 \int_0^T \|\rho\rho_x\|^2 \, \mathrm{d}t \leqslant c\|\rho_0\|_4^4. \tag{2}$$

So, from the second equation of $(1')$ and (2) it follows

$$\max_{0 \leqslant t \leqslant T} \|\partial_3 p(t)\|_4 \leqslant c\|\rho_0\|_4. \tag{3}$$

Taking the scalar product of the first equation of $(1')$ and $\hat{\mathbf{u}}$, after some trivial transformations and the use of (2) one gets

$$\max_{0 \leqslant t \leqslant T} \|\hat{\mathbf{u}}(t)\|^2 + \nu \int_0^T \|\hat{\mathbf{u}}_x\|^2 \, \mathrm{d}t \leqslant c\|\hat{\mathbf{u}}_0\|^2 + cT \equiv c. \tag{4}$$

Now it is possible to obtain an a priori estimate for the pressure function p. The main idea is to represent p as $p = p_1 + p_2$, where p_2, $\int_0^1 p_2 \, \mathrm{d}z = 0$, is an antiderivative of p_z in z and p_1, $(p_1^3, 1) = 0$, is a function of two variables (x and y) only. For the norm of p_2 the estimate $\|p_2\|_4 \leqslant c\|\partial_3 p_2\|_4 \leqslant c$ follows from (3). To estimate the norm of p_1, let us take the scalar product of the first equation of $(1')$ and $\nabla'(\Delta')^{-1} p_1^3$; here we use the Neumann boundary conditions to invert Δ'. Since p_1 is a function of the x and y variables only, it becomes possible to estimate the scalar product properly. Namely, we represent the scalar product as an iterated integral in z and Ω', apply the Hölder and multiplicative inequalities for the 2D case and then use that p_1 does not depend on z. As for estimating the integral over the boundary $\partial\Omega'$, we differentiate the first boundary condition in the tangent direction and use the second boundary condition. So, this integral is transformed to the form containing derivatives of the normal components which admits proper estimation. It should be noted that in the case when Ω' is a polygon the integral over $\partial\Omega'$ vanishes. Thus, the estimate for p is of the form

$$\max_{0\leqslant t\leqslant T}\|p_2\|_4\leqslant c\|\rho_0\|_4,\quad \|p_1\|_4\leqslant c\big(\|v_x\|^{1/2}+\|v\|^{1/2}+1\big)\|v\|^{1/2},\tag{5}$$

where $v=\hat{\mathbf{u}}^2$.

The next a priori estimate is obtained after taking the scalar product of the first equation of $(1')$ and $\hat{\mathbf{u}}^2\hat{\mathbf{u}}$:

$$\frac{1}{4}\frac{\mathrm{d}}{\mathrm{d}t}\big\|\hat{\mathbf{u}}(t)\big\|_4^4-\nu\big(\Delta\hat{\mathbf{u}},\hat{\mathbf{u}}^2\hat{\mathbf{u}}\big)+\big(\nabla'p,\hat{\mathbf{u}}^2\hat{\mathbf{u}}\big)+\big(u_k\hat{\mathbf{u}}_{x_k},\hat{\mathbf{u}}^2\hat{\mathbf{u}}\big)=0.\tag{6}$$

The last scalar product is equal to zero and the first one gives $-(\Delta\hat{\mathbf{u}},\hat{\mathbf{u}}^2\hat{\mathbf{u}})=\int_\Omega v|\nabla\hat{\mathbf{u}}|^2\,\mathrm{d}x+\frac{1}{2}\|v_x\|^2$. As for the remaining scalar product, it can be estimated as

$$\big|\big(\nabla'p,\hat{\mathbf{u}}^2\hat{\mathbf{u}}\big)\big|\leqslant\frac{\nu}{2}\|v_x\|^2+c\big(\|v\|^2\|\hat{\mathbf{u}}_x\|^2+\|v\|^2\|\hat{\mathbf{u}}_x\|+\|v\|^2+\|\hat{\mathbf{u}}_x\|^2\big).\tag{7}$$

Using these estimates and Gronwall's inequality, from (6) one gets

$$\max_{0\leqslant t\leqslant T}\big\|\hat{\mathbf{u}}(t)\big\|_4^4+\int_0^T\int_\Omega\hat{\mathbf{u}}^2|\hat{\mathbf{u}}_x|^2\,\mathrm{d}x\,\mathrm{d}t\leqslant c.\tag{8}$$

From the estimates obtained, it follows that $\int_0^T\|p\|_4^4\,\mathrm{d}t\leqslant c$.

Differentiating $(1')$ in z and using the same technique and the previous relations, it is possible to get the new estimates

$$\max_{0\leqslant t\leqslant T}\big\|\hat{\mathbf{u}}_z(t)\big\|_4^4+\nu\int_0^T\int_\Omega|\hat{\mathbf{u}}_z|^2\big(\hat{\mathbf{u}}_{zx}\big)^2\,\mathrm{d}x\,\mathrm{d}t\leqslant c,\tag{9}$$

$$\max_{0\leqslant t\leqslant T}\|\rho_z\|^2+\int_0^T\|\rho_{zx}\|^2\,\mathrm{d}t\leqslant c.\tag{10}$$

The final step in obtaining a priori estimates consists in differentiating $(1')$ in t. Using similar techniques and the estimates obtained, we have

$$\max_{0\leqslant t\leqslant T}\big(\big\|\hat{\mathbf{u}}_t(t)\big\|^2+\big\|\rho_t(t)\big\|^2\big)+\int_0^T\big(\|\hat{\mathbf{u}}_{tx}\|^2+\|\rho_{tx}\|^2\big)\,\mathrm{d}t\leqslant c_T\big(\|\hat{\mathbf{u}}_0\|_{W_2^2}^4+\|\rho_0\|_{W_2^2}^2\big).\tag{11}$$

Taking into account all the obtained a priori estimates, we get the overall estimate

$$\begin{aligned}&\max_{0\leqslant t\leqslant T}\big(\|\rho\|_4+\|\rho_z\|+\|\hat{\mathbf{u}}_x\|+\|u_3\|+\|\hat{\mathbf{u}}_z\|_4+\|\hat{\mathbf{u}}_t\|+\|\rho_t\|\big)\\&\quad+\int_0^T\big(\|\rho_x\|^2+\|\rho_{zx}\|^2+\|\rho_{tx}\|^2+\|\hat{\mathbf{u}}_x\|^2+\|\hat{\mathbf{u}}\hat{\mathbf{u}}_x\|^2+\|\hat{\mathbf{u}}_{zx}\|^2+\|\hat{\mathbf{u}}_z\hat{\mathbf{u}}_{zx}\|^2+\|\hat{\mathbf{u}}_{tx}\|^2\big)\,\mathrm{d}t\\&\leqslant c_T\big(\|\hat{\mathbf{u}}_0\|_{W_2^2}^4+\|\rho_0\|_{W_2^2}^2\big).\end{aligned}\tag{12}$$

Let us now proceed to the proof of existence and uniqueness of a solution. Introduce the following spaces:

- $\mathbf{V}_2$—a space of vector functions $\hat{\mathbf{v}}=(v_1,v_2)$ from $\mathbf{W}_2^1(Q_T)$, satisfying the boundary conditions $(1'')$ and such that $\hat{\mathbf{v}}_z\in\mathbf{W}_2^1(Q_T)$ and $\int_0^1\operatorname{div}'\hat{\mathbf{v}}(x',z,t)\,\mathrm{d}z=0$;
- R—a space of functions $r\in W_2^1(Q_T)$ such that $r_z\in W_2^1(Q_T)$.

As for the relation for ρ, take the scalar product of the last equation of $(1')$ and $r \in R$ and integrate the result in t from 0 to t. Integrating by parts gives

$$\int\limits_{Q_t} (-\rho r_t + \nu_1 \rho_x r_x - u_k \rho r_{x_k})\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega} \rho r|_{t=t}\,\mathrm{d}x - \int\limits_{\Omega} \rho_0 r|_{t=0}\,\mathrm{d}x = 0. \tag{13}$$

Taking then the scalar product of the first equation of $(1')$ and $\hat{\mathbf{v}} \in \mathbf{V}_2$ and further integration in t give

$$\int\limits_{Q_t} (-\hat{\mathbf{u}}\hat{\mathbf{v}}_t + \nu \hat{\mathbf{u}}_x \hat{\mathbf{v}}_x + l\hat{\mathbf{u}}\hat{\mathbf{v}} + \nabla' p \cdot \hat{\mathbf{v}} + u_k \hat{\mathbf{u}}_{x_k} \hat{\mathbf{v}})\,\mathrm{d}x\,\mathrm{d}t + \int\limits_{\Omega} \hat{\mathbf{u}}\hat{\mathbf{v}}|_{t=t}\,\mathrm{d}x - \int\limits_{\Omega} \hat{\mathbf{u}}_0 \hat{\mathbf{v}}|_{t=0}\,\mathrm{d}x = 0; \tag{14}$$

here u_3 is determined from $\hat{\mathbf{u}}$ via the relations $\operatorname{div} \mathbf{u} = 0$, $u_3(t, x', 0) = 0$.

It is possible to transform (14) to exclude the vertical component of the velocity as well as the pressure function:

$$\int\limits_{Q_t} \left(-\hat{\mathbf{u}}\hat{\mathbf{v}}_t + \nu \hat{\mathbf{u}}_x \hat{\mathbf{v}}_x + l\hat{\mathbf{u}}\hat{\mathbf{v}} - g\rho \int\limits_0^{x_3} \operatorname{div}' \hat{\mathbf{v}}\,\mathrm{d}z - u_j \hat{\mathbf{u}}\hat{\mathbf{v}}_{x_j} + \int\limits_0^{x_3} \operatorname{div}' \hat{\mathbf{u}}\,\mathrm{d}z\, \hat{\mathbf{u}}\hat{\mathbf{v}}_{x_3} \right) \mathrm{d}x\,\mathrm{d}t$$
$$+ \int\limits_{\Omega} \hat{\mathbf{u}}\hat{\mathbf{v}}|_{t=t}\,\mathrm{d}x - \int\limits_{\Omega} \hat{\mathbf{u}}_0 \hat{\mathbf{v}}|_{t=0}\,\mathrm{d}x = 0. \tag{15}$$

Thus, a *weak solution* to $(1')$, $(1'')$ is a pair of functions $\hat{\mathbf{u}} \in \mathbf{V}_2$, $\rho \in R$ satisfying for all $\hat{\mathbf{v}} \in \mathbf{V}_2$, $r \in R$ and arbitrary $t \in [0, T]$ the relations (13), (15).

Using (12), it is not difficult to prove uniqueness of a solution.

The existence of a solution follows from the results of [2,8], where existence of the local in time solution in the 3D case has been proven, and the estimate (12). Continuity of the norm $\|\hat{\mathbf{u}}_x\|$ in t follows from (12) and the imbedding of W_2^1 into C in 1D case.

Thus, the following statement has been proven:

Theorem. *Let $\hat{\mathbf{u}}_0 \in \mathbf{W}_2^2(\Omega)$, $\rho_0 \in W_2^2(\Omega)$, $\int_0^1 \operatorname{div}' \hat{\mathbf{u}}_0\,\mathrm{d}z = 0$ and $\hat{\mathbf{u}}_0$ satisfies the boundary conditions $(1'')$. Then for any ν, $\nu_1 > 0$ and any arbitrary $T > 0$ the problem $(1')$, $(1'')$ has in Q_T a unique weak solution such that $\hat{\mathbf{u}}^2, \hat{\mathbf{u}}_z^2, \hat{\mathbf{u}}_x, \hat{\mathbf{u}}_{zx}^2, \hat{\mathbf{u}}_t, \hat{\mathbf{u}}_{tx} \in \mathbf{L}_2(Q_T)$, and $\rho^2, \rho_x, \rho_{zx}, \rho_{tx} \in L_2(Q_T)$. The norm $\|\hat{\mathbf{u}}_x\|$ is continuous in t.*

A similar result but with a totally different method has been recently obtained in [1] under the assumption of a smooth boundary $\partial\Omega'$.

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