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On the Hochschild homology of quantum $SL(N)$

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Abstract

We show that the quantized coordinate ring $A := k_q[SL(N)]$ satisfies van den Bergh's analogue of Poincaré duality for Hochschild (co)homology with dualizing bimodule being A_σ, the A-bimodule which is A as k-vector space with right multiplication twisted by the modular automorphism σ of the Haar functional. This implies that $\mathrm{H}_{N^2-1}(A, A_\sigma) \cong k$, generalizing our previous result for $k_q[SL(2)]$. ***To cite this article: T. Hadfield, U. Krähmer, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Résumé

Sur l'homologie de Hochschild de quantum $SL(N)$. Nous démontrons que l'anneau standard quantique des coordonnées $A := k_q[SL(N)]$ satisfait l'analogue de van den Bergh de la dualité de Poincaré dans l'(co)homologie de Hochschild. Le bimodule de la dualité est A_σ, le A-bimodule qui est A comme un espace vectoriel, avec la multiplication à droite tordue par l'automorphisme modulaire σ de la fonctionnelle de Haar. Ceci implique $\mathrm{H}_{N^2-1}(A, A_\sigma) \cong k$, et généralise notre résultat précédent pour $k_q[SL(2)]$. ***Pour citer cet article : T. Hadfield, U. Krähmer, C. R. Acad. Sci. Paris, Ser. I 343 (2006).***

Version française abrégée

La dimension d'une variété régulière affine V sur un corps algébriquement clos k de caractéristique nulle est, par le théorème de Hochschild–Kostant–Rosenberg [11], égale à $\sup\{n \geqslant 0 \mid \mathrm{HH}_n(k[V]) \neq 0\}$, où $\mathrm{HH}_n(k[V])$ est l'homologie de Hochschild de l'anneau des coordonnées $k[V]$. Mais, même pour les anneaux non-commutatifs qui se comportent le mieux, l'homologie de Hochschild est souvent assez dégénérée. Par exemple, l'anneau standard quantique des coordonnées $A := k_q[SL(N)]$ est, pour les valeurs de q génériques, Auslander-régulier et Cohen–Macauley, de dimension globale et dimension de Gelfand–Kirillov égales à la dimension classique $N^2 - 1$ de $SL(N)$, mais $\mathrm{HH}_n(A) = 0$ pour $n \geqslant N$ [6].

Dans cette Note, pour $A = k_q[SL(N)]$ nous surmontons cette « chute de la dimension » en prenant des coefficients dans un bimodule approprié. La structure d'algèbre de Hopf cosemisimple sur A détermine la fonctionnelle de Haar

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$h : A \to k$, qui est invariante à droite et à gauche par le coproduit de A. De plus, il y a un unique automorphisme $\sigma \in \mathrm{Aut}(A)$, nommé automorphisme modulaire, tel que $h(xy) = h(y\sigma(x))$ pour tous $x, y \in A$ (voir [13], Section 11.3). Dans ce cas, le bimodule des coefficients crucial est A_σ, qui est A vu comme un espace vectoriel avec la structure de bimodule donnée par $x \triangleright y \triangleleft z := xy\sigma(z)$. Le théorème principal s'énonce ainsi :

Théorème 0.1. *Il existe un isomorphisme* (*d'espaces vectoriels*) $H_{N^2-1}(A, A_\sigma) \cong k$.

Nous avons démontré le cas $N = 2$ par un calcul explicite [10]. La démonstration pour N quelconque utilise l'analogue suivant de la dualité de Poincaré pour l'(co)homologie de Hochschild, démontré par van den Bergh :

Théorème 0.2. ([19]) *Soit X une k-algèbre lisse telle qu'il existe $d_X \in \mathbb{N}$ avec $H^n(X, X^e) = 0$ pour $n \neq d_X$, et de plus $U_X = H^{d_X}(X, X^e)$ est un X-bimodule inversible. Alors, pour tout X-bimodule M on a un isomorphisme de k-modules* :

$$\mathrm{H}^n(X, M) \cong \mathrm{H}_{d_X-n}(X, U_X \otimes_X M). \tag{1}$$

Ici $X^e := X \otimes X^{\mathrm{op}}$ est l'algèbre enveloppante de X (partout dans cette Note $\otimes$ désigne le produit tensoriel $\otimes_k$) et $\mathrm{H}_n(X, M) := \mathrm{Tor}_n^{X^e}(M, X)$, $\mathrm{H}^n(X, M) := \mathrm{Ext}^n_{X^e}(X, M)$ sont les groupes d'homologie et de cohomologie de Hochschild de X à coefficients dans un X-bimodule M. Nous dirons qu'une algèbre X est lisse ([19] erratum) si elle est de dimension projective finie (comme un X^e-module). Van den Bergh a prouvé que X est lisse si et seulement si X^e est de dimension globale finie. Nous dirons qu'une algèbre satisfait la condition de dualité de Poincaré si elle satisfait les hypothèses du Théorème 0.2.

Le résultat technique principal de cette Note est la démonstration du fait que le Théorème 0.2 est valide pour les anneaux standards quantiques de coordonnées $B := k_q[M(N)]$ des matrices quantiques, $C := k_q[GL(N)]$ et $A = k_q[SL(N)]$. Alors, le Théorème 0.1 suit du fait bien connu que le centre de A est réduit aux scalaires.

Notre motivation principale pour cette étude des $H_*(A, A_\sigma)$ est la cohomologie dite cyclique tordue et ses relations avec les calculs différentiels covariants, les deux étant dûs à Kustermans, Murphy et Tuset [14]. Cette théorie cyclique (dans le sens de Connes [4]) conditionne un couple formé d'une algèbre X et d'un automorphisme σ, et elle contient l'homologie simpliciale sous-jacente $H_*(X, X_\sigma)$ (au moins dans le cas où σ est diagonalisable, voir Proposition 2.1 de [10]). Les formes volumiques des calculs différentiels covariants définissent des cocycles cycliques tordus, et l'apparition d'un automorphisme de torsion est forcé par les propriétés modulaires de la fonctionnelle de Haar, qui remplace les traces de la construction originale de Connes [4]. Nous voyons de part le Théorème 0.2 que les coefficients tordus se présentent naturellement pour des raisons purement homologiques, et de plus le Théorème 0.1 et quelques résultats similaires pour les hyperplans quantiques et les sphères quantiques de Podleś [9,18] démontrent que la torsion détermine, comme dans le cas classique, une classe unique de degré le plus haut dans l'homologie de Hochschild.

1. Introduction and statement of the result

According to the Hochschild–Kostant–Rosenberg theorem [11], the dimension of a regular affine variety V over an algebraically closed field k of characteristic 0 can be expressed in terms of the Hochschild homology of its coordinate ring $k[V]$ as

$$\dim(V) = \sup\bigl\{n \geqslant 0 \mid \mathrm{HH}_n\bigl(k[V]\bigr) \neq 0\bigr\}. \tag{2}$$

However, even for well-behaved noncommutative algebras Hochschild homology is often rather degenerate. For example, the standard quantized coordinate ring $A := k_q[SL(N)]$ is for generic q Auslander regular and Cohen–Macaulay with global and Gelfand–Kirillov dimension equal to the classical dimension $N^2 - 1$ of $SL(N)$ [15], but $\mathrm{HH}_n(A) = 0$ for $n \geqslant N$ [6]. In this Note we show that this 'dimension drop' is overcome by passing to Hochschild homology $\mathrm{H}_*(A, M)$ with coefficients in a suitable bimodule M.

The cosemisimple Hopf algebra structure on A determines the Haar functional $h : A \to k$ which is left and right invariant under the coaction of A on itself, and there is a unique automorphism $\sigma \in \mathrm{Aut}(A)$, the so-called modular automorphism, such that $h(xy) = h(\sigma(y)x)$ for all $x, y \in A$ (see [13], Section 11.3). The crucial coefficient bimodule M is then A_σ which is A as k-vector space with bimodule structure $x \triangleright y \triangleleft z := xy\sigma(z)$. Our main result is:

Theorem 1.1. *There is an isomorphism of k-vector spaces* $\mathrm{H}_{N^2-1}(A, A_\sigma) \cong k$.

For $N = 2$ this was shown by explicit calculation in [10]. The proof for arbitrary N given below relies on the following analogue of Poincaré duality for Hochschild (co)homology proven by van den Bergh:

Theorem 1.2. ([19]) *Let X be a smooth algebra such that there exists $d_X \in \mathbb{N}$ with $\mathrm{H}^n(X, X^e) = 0$ for $n \neq d_X$, and that $U_X := \mathrm{H}^{d_X}(X, X^e)$ is an invertible X-bimodule. Then for every X-bimodule M we have*

$$\mathrm{H}^n(X, M) \cong \mathrm{H}_{d_X - n}(X, U_X \otimes_X M). \tag{3}$$

Here $X^e := X \otimes X^{\mathrm{op}}$ is the enveloping algebra of X (throughout this paper an unadorned $\otimes$ means tensor product over k), so the Hochschild homology and cohomology groups of X with coefficients in M are $\mathrm{H}_n(X, M) = \mathrm{Tor}_n^{X^e}(M, X)$ and $\mathrm{H}^n(X, M) = \mathrm{Ext}^n_{X^e}(X, M)$, respectively. Following [19] (erratum) an algebra X is called smooth if it has finite projective dimension $\mathrm{pd}_{X^e}(X) = \inf\{n \geqslant 0 \mid \mathrm{H}^{n+1}(X, \cdot) = 0\}$ as an X^e-module. As in [3] we call $\mathrm{pd}_{X^e}(X)$ the dimension of X and denote it by $\dim(X)$. As pointed out by van den Bergh, X is smooth if and only if X^e has finite global dimension. This follows from $\mathrm{gldim}(X) \leqslant \dim(X) \leqslant \mathrm{gldim}(X^e)$ and $\dim(X \otimes Y) \leqslant \dim(X) + \dim(Y)$ ([3], Propositions IX.7.4–IX.7.6), which gives $\mathrm{gldim}(X^e) \leqslant \dim(X^e) \leqslant 2\dim(X) \leqslant 2\,\mathrm{gldim}(X^e)$. In the sequel we say that an algebra has the Poincaré duality property if it satisfies the assumptions of Theorem 1.2.

The principal technical result of this Note consists in remarking successively that Theorem 1.2 applies to the quantized coordinate rings $B := k_q[M(N)]$ of $N \times N$-matrices, $C := k_q[GL(N)]$ and $A = k_q[SL(N)]$. Theorem 1.1 then follows from the well-known fact that the center of A consists only of the scalars.

Our main motivation for studying $\mathrm{H}_*(A, A_\sigma)$ is the so-called twisted cyclic cohomology and its link to covariant differential calculi over quantum groups both due to Kustermans, Murphy and Tuset [14]. Twisted cyclic cohomology is defined by a cyclic object in the sense of Connes [4] depending on an algebra X and an automorphism σ. Its underlying simplicial homology is $\mathrm{H}_*(X, X_\sigma)$ (at least when σ is diagonalizable, see Proposition 2.1 in [10]). The volume forms of covariant differential calculi over quantum groups define twisted cyclic cocycles, with the appearance of the twisting automorphism forced by the modular properties of the Haar functional that replaces the traces of Connes' original construction [4]. In view of Theorem 1.2 twisted coefficients appear very naturally also for purely homological reasons, and Theorem 1.1 and similar results for quantum hyperplanes and Podleś quantum spheres [9, 18] show that the twist determines as in the classical case a unique class of top degree in Hochschild homology.

2. Proof of Theorem 1.1

We first consider the quantized coordinate ring $B = k_q[M(N)]$. Recall that this has generators u_{ij}, $1 \leqslant i, j \leqslant N$, with relations

$$\begin{aligned} &u_{ik}u_{il} = qu_{il}u_{ik}, \quad u_{ik}u_{jk} = qu_{jk}u_{ik}, \quad u_{ik}u_{jk} = qu_{jk}u_{ik}, \\ &u_{jk}u_{jl} = qu_{jl}u_{jk}, \quad u_{il}u_{jk} = u_{jk}u_{il}, \quad u_{ik}u_{jl} - u_{jl}u_{ik} = \left(q - q^{-1}\right)u_{il}u_{jk} \end{aligned} \tag{4}$$

for all $i < j$, $k < l$. Here $q \in k \setminus \{0\}$ is a fixed deformation parameter, assumed not to be a root of unity.

Proposition 2.1. *B has the Poincaré duality property with $d_B = N^2$ and $U_B = B_\sigma$, with σ defined by*

$$\sigma(u_{ij}) := q^{2(N+1-i-j)}u_{ij}. \tag{5}$$

We will use here and later the following Künneth-type isomorphism of Cartan and Eilenberg:

Theorem 2.2. ([3], Theorem XI.3.1) *Let k be a field, A_1, A_2 be two left Noetherian k-algebras and M_i, N_i be finitely generated left modules over A_i. Then*

$$\bigoplus_{i+j=n} \mathrm{Ext}^i_{A_1}(M_1, N_1) \otimes \mathrm{Ext}^j_{A_2}(M_2, N_2) \cong \mathrm{Ext}^n_{A_1 \otimes A_2}(M_1 \otimes M_2, N_1 \otimes N_2). \tag{6}$$

Proof of Proposition 2.1. The claim follows from Proposition 2 in [19]: As mentioned in [16] it follows from a result of Priddy ([17], Theorem 5.3) that B is a graded Koszul algebra. By definition the Koszul dual $B^!$ has generators $\hat{u}_{ij}$ with relations orthogonal to (4):

$$\hat{u}_{ij}^2 = 0 \quad \forall i, j, \quad \hat{u}_{ik}\hat{u}_{il} = -q^{-1}\hat{u}_{il}\hat{u}_{ik}, \quad \hat{u}_{ik}\hat{u}_{jk} = -q^{-1}\hat{u}_{jk}\hat{u}_{ik}, \quad \hat{u}_{ik}\hat{u}_{jk} = -q^{-1}\hat{u}_{jk}\hat{u}_{ik},$$
$$\hat{u}_{jk}\hat{u}_{jl} = -q^{-1}\hat{u}_{jl}\hat{u}_{jk}, \quad \hat{u}_{ik}\hat{u}_{jl} = -\hat{u}_{jl}\hat{u}_{ik}, \quad \hat{u}_{il}\hat{u}_{jk} + \hat{u}_{jk}\hat{u}_{il} = (q^{-1} - q)\hat{u}_{ik}\hat{u}_{jl}, \tag{7}$$

where $i < j, k < l$. These relations imply that the monomials $\hat{u}_{i_1 j_1} \cdots \hat{u}_{i_n j_n}$, $n = 1, \ldots, N^2$, $i_1 j_1 \prec \ldots \prec i_n j_n$ with respect to lexicographical ordering, form a k-linear basis, and that $B^!$ is Frobenius with Frobenius functional $\hat{h} : B^! \to k$ being projection onto the component of the longest basis element $\hat{u}_{11}\hat{u}_{12} \cdots \hat{u}_{NN-1}\hat{u}_{NN}$ (that is, for each nonzero $x \in B^!$ there exists $y \in B^!$ with $\hat{h}(xy) \neq 0$). The formula for σ follows by straightforward computation using the relations (7).

Smoothness of B follows from some well-known facts about Koszul algebras (see, e.g., the survey [7]). First, $\mathrm{Tor}_n^{B^e}(k,k) \cong \mathrm{Ext}_{B^e}^n(k,k)$, and by Theorem 2.2 and Koszulity this can be written as $\sum_{i+j=n} B_i^! \otimes B_j^!$ (note that $B \cong B^{\mathrm{op}}$). Thus $\mathrm{Tor}_n^{B^e}(k,k) = 0$ for $n > 2N^2$, hence $\dim(B) \leqslant 2N^2$ by [1], Corollary 8.7.5. □

It was shown by Farinati that the class of algebras having the Poincaré duality property is closed under localization [5], Theorem 1.5. The quantized coordinate ring $C = k_q[GL(N)]$ of the general linear group is the localization of $B = k_q[M(N)]$ at the central quantum determinant [16]

$$\det_q = \sum_{\pi \in S_N} (-q)^{l(\pi)} u_{1\pi(1)} \cdots u_{N\pi(N)}, \tag{8}$$

with S_N the permutation group on N elements and $l(\pi)$ the length of a permutation. This is σ-invariant, so σ passes to an automorphism of C, still denoted by σ and given by (5). Proposition 2.1 now implies:

Corollary 2.3. *C has the Poincaré duality property with $d_C = d_B = N^2$ and $U_C = C \otimes_B U_B = C_\sigma$.*

The algebra $A = k_q[SL(N)]$ is the quotient of B by the relation $\det_q = 1$, and again by σ-invariance of $\det_q$, σ descends to an automorphism of A. Following the strategy of Levasseur and Stafford [15] we will use the isomorphism $C \cong A \otimes D$, where $D := k[t, t^{-1}]$ to deduce Poincaré duality for A from B via C. This enables us to prove finally:

Proposition 2.4. *The algebra A has the Poincaré duality property with $d_A = N^2 - 1$ and $U_A = A_\sigma$.*

Proof. We apply Theorem 2.2 with $A_1 = N_1 = A^e$, $M_1 = A$ and $A_2 = N_2 = D^e$, $M_2 = D$. Since $A = A^{\mathrm{op}}$ (the antipode of the standard Hopf algebra structure gives an isomorphism) we have $A^e \cong k_q[SL(N) \times SL(N)]$, so it is (both left and right) Noetherian by [12], Proposition 9.2.2 and further A is smooth by [8]. It is elementary to show that D satisfies Poincaré duality with $d_D = 1$, $U_D = D$. So $\mathrm{Ext}_{A^e}^n(A, A^e) \otimes D \cong \mathrm{Ext}_{C^e}^{n+1}(C, C^e)$ for each $n \geqslant 0$. So by Corollary 2.3 we have $\mathrm{Ext}_{A^e}^{N^2-1}(A, A^e) \otimes D = C_\sigma$, which is $A_\sigma \otimes D$, and all other $\mathrm{Ext}_{A^e}^n(A, A^e)$ vanish. The result follows. □

Thus there is an isomorphism $\mathrm{H}_{N^2-1}(A, A_\sigma) \cong \mathrm{H}^0(A, A)$. The latter is by definition the center of A, and this consists only of the scalars (see, e.g., [12], Theorem 9.3.20). This completes the proof of Theorem 1.1.

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