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Mathematical Problems in Mechanics

New formulations of linearized elasticity problems, based on extensions of Donati's theorem

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Abstract

The classical Donati theorem is used for characterizing smooth matrix fields as linearized strain tensor fields. In this Note, we give several generalizations of this theorem, notably to matrix fields whose components are only in H^{-1}. We then show that our extensions of Donati's theorem allow to reformulate in a novel fashion linearized three-dimensional elasticity problems as quadratic minimization problems with the strains as the primary unknowns. ***To cite this article: C. Amrouche et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Nouvelles formulations de problèmes d'élasticité linéarisée, basées sur des généralisations du théorème de Donati. Le théorème classique de Donati sert à caractériser les champs de matrices réguliers qui sont des champs de déformation linéarisés. Dans cette Note, on donne plusieurs généralisations de ce théorème, en particulier à des champs de matrices dont les composantes sont seulement dans H^{-1}. On montre ensuite que de telles généralisations conduisent à de nouvelles formulations des problèmes d'élasticité linéarisée tridimensionnelle, comme des problèmes de minimisation quadratique où les déformations sont les inconnues principales. ***Pour citer cet article : C. Amrouche et al., C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Notations and preliminaries

Latin indices vary in the set $\{1, 2, 3\}$ and the summation convention with respect to repeated indices is used in conjunction with this rule.

Let Ω be an open subset of $\mathbb{R}^3$ and let $x = (x_i)$ designate a generic point in Ω. Partial derivative operators of the first, second, and third order are then denoted $\partial_i := \partial/\partial x_i$, $\partial_{ij} := \partial^2/\partial x_i \partial x_j$, and $\partial_{ijk} := \partial^3/\partial x_i \partial x_j \partial x_k$. Spaces

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of functions, vector fields, and matrix fields, defined over Ω are respectively denoted by italic capitals, boldface Roman capitals, and special Roman capitals. The subscript s appended to a special Roman capital denotes a space of symmetric matrix fields. The notation $D(\Omega)$ denotes the space of functions that are infinitely differentiable in Ω and have compact supports in Ω. The notation $D'(\Omega)$ denotes the space of distributions defined over Ω.

A *domain* in $\mathbb{R}^3$ is a bounded, connected, open subset of $\mathbb{R}^3$ with a Lipschitz-continuous boundary.

The detailed proofs of the results announced in this Note are given in [2].

2. Extensions of Donati's theorem

Let Ω be an open subset in $\mathbb{R}^3$. For any vector field $\mathbf{v} = (v_i) \in \mathbf{D}'(\Omega)$, the *linearized strain tensor field* associated with the vector field $\mathbf{v}$ is the symmetric matrix field $\boldsymbol{\nabla}_s\mathbf{v} \in \mathbb{D}'_s(\Omega)$ defined by

$$\boldsymbol{\nabla}_s\mathbf{v} := \frac{1}{2}\left(\boldsymbol{\nabla}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{\nabla}\boldsymbol{v}\right),$$

or equivalently, by $(\boldsymbol{\nabla}_s\mathbf{v})_{ij} = \frac{1}{2}(\partial_i v_j + \partial_j v_i)$. When Ω is connected, the kernel of the operator $\boldsymbol{\nabla}_s$ has the well-known characterization

$$\mathbf{Ker}\,\boldsymbol{\nabla}_s = \left\{\mathbf{v} \in \mathbf{D}'(\Omega);\ \boldsymbol{\nabla}_s\mathbf{v} = \mathbf{0} \text{ in } \mathbb{D}'(\Omega)\right\} = \left\{\mathbf{v} = \boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega;\ \boldsymbol{a} \in \mathbb{R}^3,\ \boldsymbol{b} \in \mathbb{R}^3\right\},$$

where $\mathbf{id}_\Omega$ denotes the identity mapping of the set Ω.

A first important property of the operator $\boldsymbol{\nabla}_s$ is given in the next theorem, which constitutes the $\mathbb{H}^m_s(\Omega)$*-matrix version of J.-L. Lions' lemma*.

Theorem 2.1. *Let Ω be a domain in $\mathbb{R}^3$ and let a vector field $\mathbf{v} \in \mathbf{D}'(\Omega)$ be such that $\boldsymbol{\nabla}_s\mathbf{v} \in \mathbb{H}^m_s(\Omega)$ for some integer $m \in \mathbb{Z}$. Then $\mathbf{v} \in \mathbf{H}^{m+1}(\Omega)$.*

The next theorem lists two properties of the operator $\boldsymbol{\nabla}_s$, *considered as acting from the space* $\mathbf{L}^2(\Omega)$ *into the space* $\mathbb{H}^{-1}_s(\Omega)$. The proof relies on the $\mathbb{H}^{-1}_s(\Omega)$-matrix version of J.-L. Lions' lemma (Theorem 2.1) and the closed graph theorem, combined with an abstract result due to Peetre [9] and Tartar [10].

Theorem 2.2. *Let Ω be a domain in $\mathbb{R}^3$.*

(a) *The operator $\boldsymbol{\nabla}_s : \dot{\mathbf{L}}^2(\Omega) := \mathbf{L}^2(\Omega)/\mathbf{Ker}\,\boldsymbol{\nabla}_s \to \mathbb{H}^{-1}_s(\Omega)$, where for any $\dot{\mathbf{v}} \in \dot{\mathbf{L}}^2(\Omega)$, $\boldsymbol{\nabla}_s\dot{\mathbf{v}} := \boldsymbol{\nabla}_s\mathbf{w}$ for any $\mathbf{w} \in \dot{\mathbf{v}}$, is an isomorphism from $\dot{\mathbf{L}}^2(\Omega)$ onto $\mathbf{Im}\,\boldsymbol{\nabla}_s$. Consequently, the space $\mathbf{Im}\,\boldsymbol{\nabla}_s$ is closed in $\mathbb{H}^{-1}_s(\Omega)$.*

(b) *The dual operator of $\boldsymbol{\nabla}_s : \mathbf{L}^2(\Omega) \to \mathbb{H}^{-1}_s(\Omega)$ is $-\mathbf{div} : \mathbb{H}^1_{0,s}(\Omega) \to \mathbf{L}^2(\Omega)$ and the dual operator of $\boldsymbol{\nabla}_s : \dot{\mathbf{L}}^2(\Omega) \to \mathbb{H}^{-1}_s(\Omega)$ is $-\mathbf{div} : \mathbb{H}^1_{0,s}(\Omega) \to \dot{\mathbf{L}}^2(\Omega)$.*

The two theorems above show that, indeed, *the operator $\boldsymbol{\nabla}_s$ is the 'matrix analog' of the usual gradient operator* **grad**. More specifically, the implication established in Theorem 2.1 is the matrix analog of the 'usual' J.-L. Lions lemma (see Duvaut and Lions [5, Chapter 3]), as extended by Amrouche and Girault [3, Proposition 2.10]): *Let Ω be a domain in $\mathbb{R}^3$ and let a distribution $v \in D'(\Omega)$ be such that $\mathbf{grad}\, v \in \mathbf{H}^m(\Omega)$ for some integer $m \in \mathbb{Z}$. Then $v \in H^{m+1}(\Omega)$*. Likewise, part (a) of Theorem 2.2 mimics that **grad** is an isomorphism from $L^2(\Omega)/\mathbb{R}$ onto its image in $\mathbf{H}^{-1}(\Omega)$ (cf. Girault and Raviart [7, Corollary 2.4]) and part (b) mimics that the dual operator of $\mathbf{grad} : L^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$ is $-\operatorname{div} : \mathbf{H}^1_0(\Omega) \to L^2(\Omega)$.

The next theorem lists two properties of the operator $\boldsymbol{\nabla}_s$, now *considered as acting from* $\mathbf{H}^1_0(\Omega)$ *into* $\mathbb{L}^2_s(\Omega)$. Its proof is similar to that of parts (a) and (b) of Theorem 2.2, and actually simpler since $\mathbf{Ker}\,\boldsymbol{\nabla}_s = \{\mathbf{0}\}$ in this case.

Theorem 2.3. *Let Ω be a domain in $\mathbb{R}^3$.*

(a) *The operator $\boldsymbol{\nabla}_s : \mathbf{H}^1_0(\Omega) \to \mathbb{L}^2_s(\Omega)$ is an isomorphism from $\mathbf{H}^1_0(\Omega)$ onto $\mathbf{Im}\,\boldsymbol{\nabla}_s$. Consequently, the space $\mathbf{Im}\,\boldsymbol{\nabla}_s$ is closed in $\mathbb{L}^2_s(\Omega)$.*

(b) *The dual operator of $\boldsymbol{\nabla}_s : \mathbf{H}^1_0(\Omega) \to \mathbb{L}^2_s(\Omega)$ is $-\mathbf{div} : \mathbb{L}^2_s(\Omega) \to \mathbf{H}^{-1}(\Omega)$.*

The operator $\boldsymbol{\nabla}_s$ can also be considered as *acting from* $\mathbf{H}^1(\Omega)$ *into* $\mathbb{L}^2_s(\Omega)$, in which case similar arguments show that the operator $\boldsymbol{\nabla}_s : \mathbf{H}^1(\Omega)/\mathbf{Ker}\,\boldsymbol{\nabla}_s \to \mathbf{Im}\,\boldsymbol{\nabla}_s$ is an *isomorphism*, so that $\mathbf{Im}\,\boldsymbol{\nabla}_s$ is again a *closed* subspace of $\mathbb{L}^2_s(\Omega)$.

Interestingly, under the additional assumption that Ω is *simply-connected*, the space $\mathbf{Im}\,\boldsymbol{\nabla}_s$ can be given an *explicit characterization* in this case, as

$$\mathbf{Im}\,\boldsymbol{\nabla}_s = \left\{\mathbf{e} = (e_{ij}) \in \mathbb{L}^2_s(\Omega);\ \partial_{lj} e_{ik} + \partial_{ki} e_{jl} - \partial_{li} e_{jk} - \partial_{kj} e_{il} = 0 \text{ in } H^{-2}(\Omega)\right\},$$

thus providing another proof that $\mathbf{Im}\,\boldsymbol{\nabla}_s$ is closed in $\mathbb{L}^2_s(\Omega)$ when the operator $\boldsymbol{\nabla}_s$ is considered as acting from $\mathbf{H}^1(\Omega)$ into $\mathbb{L}^2_s(\Omega)$. This characterization of $\mathbf{Im}\,\boldsymbol{\nabla}_s$ is due to Ciarlet and Ciarlet, Jr. [4] (for a different, and simpler, proof, see [1] or [2]).

In what follows, **CURL** designates the *matrix curl operator* and $\mathbf{e} : \mathbf{s}$ designates the *matrix inner-product* of two matrices $\mathbf{e}$ and $\mathbf{s}$ of order three.

A.J.C.B. de Saint Venant announced in 1864 what is since then known as *Saint Venant's theorem*: Let Ω be a simply-connected open subset of $\mathbb{R}^3$. Given a matrix field $\mathbf{e} \in \mathbb{C}^2_s(\Omega)$, there exists a vector field $\mathbf{v} \in \mathbf{C}^3(\Omega)$ such that $\mathbf{e} = \boldsymbol{\nabla}_s \mathbf{v}$ in Ω if $\mathbf{CURL\,CURL}\,\mathbf{e} = \mathbf{0}$ in Ω.

Then in 1890, L. Donati proved that, if Ω is an open subset of $\mathbb{R}^3$ and a matrix field $\mathbf{e} \in \mathbb{C}^2_s(\Omega)$ satisfies $\int_\Omega \mathbf{e} : \mathbf{s}\,\mathrm{d}x = 0$ for all $\mathbf{s} \in \mathbb{D}_s(\Omega)$ such that $\mathbf{div}\,\mathbf{s} = \mathbf{0}$ in Ω, then $\mathbf{CURL\,CURL}\,\mathbf{e} = 0$ in Ω. This result, known as *Donati's theorem*, thus provides, once combined with Saint Venant's theorem, another characterization of symmetric matrix fields as linearized strain tensor fields, at least for simply-connected open subsets Ω of $\mathbb{R}^3$.

Donati's theorem was generalized as follows by Ting [11] in 1974: *If Ω is a domain and a tensor field $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$ satisfies $\int_\Omega \mathbf{e} : \mathbf{s}\,\mathrm{d}x = 0$ for all $\mathbf{s} \in \mathbb{D}_s(\Omega)$ such that $\mathbf{div}\,\mathbf{s} = \mathbf{0}$ in Ω, then there exists $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \boldsymbol{\nabla}_s \mathbf{v}$ in $\mathbb{L}^2_s(\Omega)$.*

Then Moreau [8] showed in 1979 that Donati's theorem holds even in the sense of distributions, according to the following theorem, where Ω is now an arbitrary open subset of $\mathbb{R}^3$: *If a tensor field $\mathbf{e} \in \mathbb{D}'_s(\Omega)$ satisfies ${}_{\mathbb{D}'_s(\Omega)}\langle \mathbf{e}, \mathbf{s}\rangle_{\mathbb{D}_s(\Omega)} = 0$ for all $\mathbf{s} \in \mathbb{D}_s(\Omega)$ such that $\mathbf{div}\,\mathbf{s} = \mathbf{0}$ in Ω, then there exists $\mathbf{v} \in \mathbf{D}'(\Omega)$ such that $\mathbf{e} = \boldsymbol{\nabla}_s \mathbf{v}$ in the sense of distributions.* Note that Ting's and Moreau's extensions do *not* require that Ω be simply-connected.

One objective of this Note is to provide *further extensions of Donati's theorem*. To begin with, we obtain our *first extension of Donati's theorem* as a corollary to Theorem 2.2:

Theorem 2.4. *Let Ω be a domain in $\mathbb{R}^3$ and let there be given a matrix field $\mathbf{e} \in \mathbb{H}^{-1}_s(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{e} = \boldsymbol{\nabla}_s \mathbf{v}$ in $\mathbb{H}^{-1}_s(\Omega)$ if and only if*

$${}_{\mathbb{H}^{-1}_s(\Omega)}\langle \mathbf{e}, \mathbf{s}\rangle_{\mathbb{H}^1_{0,s}(\Omega)} = 0 \quad \textit{for all } \mathbf{s} \in \mathbb{H}^1_{0,s}(\Omega) \textit{ satisfying } \mathbf{div}\,\mathbf{s} = \mathbf{0} \textit{ in } \mathbf{L}^2(\Omega).$$

All other vector fields $\tilde{\mathbf{v}} \in \mathbf{L}^2(\Omega)$ satisfying $\mathbf{e} = \boldsymbol{\nabla}_s \tilde{\mathbf{v}}$ in $\mathbb{H}^{-1}_s(\Omega)$ are of the form $\tilde{\mathbf{v}} = \mathbf{v} + \boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega$ for some vectors $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$.

Proof. Since the dual operator of $\boldsymbol{\nabla}_s : \mathbf{L}^2(\Omega) \to \mathbb{H}^{-1}_s(\Omega)$ is $-\mathbf{div} : \mathbb{H}^1_{0,s}(\Omega) \to \mathbf{L}^2(\Omega)$ and the space $\mathbf{Im}\,\boldsymbol{\nabla}_s$ is closed in $\mathbb{H}^{-1}_s(\Omega)$ (Theorem 2.2), the conclusion follows from Banach's closed range theorem. That all other solutions $\tilde{\mathbf{v}}$ of the equation $\mathbf{e} = \boldsymbol{\nabla}_s \tilde{\mathbf{v}}$ are of the form indicated above follows from the characterization of the space $\mathbf{Ker}\,\boldsymbol{\nabla}_s$ recalled earlier. $\square$

This extension of Donati's theorem is the 'matrix analog' of a well-known characterization of vector fields as gradients of scalar functions (see Girault and Raviart [7, Lemma 2.1]), where the operator **grad** and the spaces $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}^1_0(\Omega)$ are replaced by their matrix analogs $\boldsymbol{\nabla}_s$ and $\mathbb{H}^{-1}_s(\Omega)$ and $\mathbb{H}^1_{0,s}(\Omega)$.

We similarly obtain a *second extension of Donati's theorem*, this time as a corollary to Theorem 2.3.

Theorem 2.5. *Let Ω be a domain in $\mathbb{R}^3$ and let there be given a matrix field $\mathbf{e} \in \mathbb{L}^2_s(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$ such that $\mathbf{e} = \boldsymbol{\nabla}_s \mathbf{v}$ in $\mathbb{L}^2_s(\Omega)$ if and only if*

$$\int_\Omega \mathbf{e} : \mathbf{s}\,\mathrm{d}x = 0 \quad \textit{for all } \mathbf{s} \in \mathbb{L}^2_s(\Omega) \textit{ satisfying } \mathbf{div}\,\mathbf{s} = 0 \textit{ in } \mathbf{H}^{-1}(\Omega).$$

If this is the case, the vector field $\mathbf{v}$ is unique.

Proof. Since the dual operator of $\nabla_s : \mathbf{H}_0^1(\Omega) \to \mathbb{L}_s^2(\Omega)$ is $-\mathbf{div} : \mathbb{L}_s^2(\Omega) \to \mathbf{H}^{-1}(\Omega)$ and the space $\mathbf{Im}\,\nabla_s$ is closed in $\mathbb{L}_s^2(\Omega)$ (Theorem 2.3), the existence of the vector field $\mathbf{v}$ again follows from Banach's closed range theorem. That $\mathbf{Ker}\,\nabla_s = \{\mathbf{0}\}$ in this case implies that such a vector field $\mathbf{v}$ is unique. □

Characterizations similar to Theorems 2.4 and 2.5 have been simultaneously obtained by Geymonat and Krasucki [6], who also noticed that Theorem 2.4 can be extended to matrix fields $\mathbf{e} \in \mathbb{W}_s^{-1,p}(\Omega)$, $1 < p < \infty$.

Finally, a *third extension of Donati's theorem* can also be obtained that also constitutes an extension of Ting [11]. The proof given here is considerably simpler, however (especially when the domain Ω is not simply-connected).

Theorem 2.6. *Let Ω be a domain in $\mathbb{R}^3$ and let there be given a matrix field $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$. Then there exists a vector field $\mathbf{v} \in \mathbf{H}^1(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$ in $\mathbb{L}_s^2(\Omega)$ if and only if*

$$\int_\Omega \mathbf{e} : \mathbf{s}\,\mathrm{d}x = 0 \quad \textit{for all } \mathbf{s} \in \mathbb{H}_{0,s}^1(\Omega) \textit{ satisfying } \mathbf{div}\,\mathbf{s} = \mathbf{0} \textit{ in } \mathbf{L}^2(\Omega).$$

All other vector fields $\mathbf{v} \in \mathbf{H}^1(\Omega)$ satisfying $\mathbf{e} = \nabla_s \mathbf{v}$ are of the form $\tilde{\mathbf{v}} = \mathbf{v} + \boldsymbol{a} + \boldsymbol{b} \wedge \mathbf{id}_\Omega$ for some vectors $\boldsymbol{a} \in \mathbb{R}^3$ and $\boldsymbol{b} \in \mathbb{R}^3$.

Proof. Let $\mathbf{e} \in \mathbb{L}_s^2(\Omega)$ be such that $\int_\Omega \mathbf{e} : \mathbf{s}\,\mathrm{d}x = 0$ for all $\mathbf{s} \in \mathbb{H}_{0,s}^1(\Omega)$ satisfying $\mathbf{div}\,\mathbf{s} = \mathbf{0}$ in $\mathbf{L}^2(\Omega)$. Since $\mathbb{L}_s^2(\Omega) \subset \mathbb{H}_s^{-1}(\Omega)$, Theorem 2.4 shows that there exists $\mathbf{v} \in \mathbf{L}^2(\Omega)$ such that $\mathbf{e} = \nabla_s \mathbf{v}$, and thus the $\mathbb{L}_s^2(\Omega)$-matrix version of J.-L. Lions' lemma (Theorem 2.1) further shows that $\mathbf{v} \in \mathbf{H}^1(\Omega)$. The other assertions are easily established. □

3. Another approach to linearized elasticity

Let Ω be a domain in $\mathbb{R}^3$, now viewed as the reference configuration of a linearly elastic body. This body is characterized by its elasticity tensor field $\mathbf{A} = (A_{ijkl})$, whose components $A_{ijkl} \in L^\infty(\Omega)$ satisfy the symmetry relations $A_{ijkl} = A_{jikl} = A_{klij}$ and are such that there exists a constant $\alpha > 0$ such that $\mathbf{A}(x)\mathbf{t} : \mathbf{t} \geqslant \alpha \mathbf{t} : \mathbf{t}$ for almost all $x \in \Omega$ and all symmetric matrices $\mathbf{t} = (t_{ij})$ of order three, where $(\mathbf{A}(x)\mathbf{t})_{ij} := A_{ijkl}(x) t_{kl}$. The body is subjected to applied body forces with density $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$. Finally, it is assumed that the linear form $L \in \mathcal{L}(\mathbf{H}^1(\Omega); \mathbb{R})$ defined by $L(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\,\mathrm{d}x$ for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$ vanishes for all $\mathbf{v} \in \mathbf{Ker}\,\nabla_s$, where ∇_s is here considered to be acting from $\mathbf{H}^1(\Omega)$ into $\mathbb{L}_s^2(\Omega)$. Then the corresponding *pure traction problem of three-dimensional linearized elasticity* classically consist in finding $\dot{\mathbf{u}} \in \dot{\mathbf{H}}^1(\Omega) := \mathbf{H}^1(\Omega)/\mathbf{Ker}\,\nabla_s$ such that

$$J(\dot{\mathbf{u}}) = \inf_{\dot{\mathbf{v}} \in \dot{\mathbf{H}}^1(\Omega)} J(\dot{\mathbf{v}}), \quad \text{where } J(\dot{\mathbf{v}}) := \frac{1}{2} \int_\Omega \mathbf{A} \nabla_s \dot{\mathbf{v}} : \nabla_s \dot{\mathbf{v}}\,\mathrm{d}x - L(\dot{\mathbf{v}}).$$

Thanks to Theorem 2.6, this problem can be recast as *another quadratic minimization problem*, this time with $\boldsymbol{\varepsilon} := \nabla_s \dot{\mathbf{u}} \in \mathbb{L}_{\mathrm{sym}}^2(\Omega)$ as the primary unknown.

Theorem 3.1. *Let Ω be a domain in $\mathbb{R}^3$. Define the Hilbert space*

$$\mathbf{E}_1(\Omega) := \left\{ \mathbf{e} \in \mathbb{L}_s^2(\Omega);\ \int_\Omega \mathbf{e} : \mathbf{s}\,\mathrm{d}x = 0 \textit{ for all } \mathbf{s} \in \mathbb{H}_{0,s}^1(\Omega) \textit{ satisfying } \mathbf{div}\,\mathbf{s} = \mathbf{0} \textit{ in } \mathbf{L}^2(\Omega) \right\},$$

and, for each $\mathbf{e} \in \mathbf{E}_1(\Omega)$, let $\mathcal{F}_1(\mathbf{e})$ denote the unique element in the quotient space $\dot{\mathbf{H}}^1(\Omega)$ that satisfies $\nabla_s \mathcal{F}_1(\mathbf{e}) = \mathbf{e}$ (Theorem 2.6). Then the mapping $\mathcal{F}_1 : \mathbf{E}_1(\Omega) \to \dot{\mathbf{H}}^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbf{E}_1(\Omega)$ and $\dot{\mathbf{H}}^1(\Omega)$. Furthermore, the minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbf{E}_1(\Omega)$ such that

$$j_1(\boldsymbol{\varepsilon}) = \inf_{\mathbf{e} \in \mathbf{E}_1(\Omega)} j_1(\mathbf{e}), \quad \textit{where } j_1(\mathbf{e}) := \frac{1}{2} \int_\Omega \mathbf{A}\mathbf{e} : \mathbf{e}\,\mathrm{d}x - L \circ \mathcal{F}_1(\mathbf{e}),$$

has one and only one solution $\boldsymbol{\varepsilon}$, and this solution satisfies $\boldsymbol{\varepsilon} = \nabla_s \dot{\mathbf{u}}$, where $\dot{\mathbf{u}}$ is the unique minimizer of the functional J in the space $\dot{\mathbf{H}}^1(\Omega)$.

Sketch of proof. By Theorem 2.6, the mapping $\mathcal{F}_1$ is a bijection between the Hilbert spaces $\mathbf{E}_1(\Omega)$ and $\dot{\mathbf{H}}^1(\Omega)$ and it can be shown that its inverse is continuous. Hence $\mathcal{F}_1 : \mathbf{E}_1(\Omega) \to \mathbf{H}^1(\Omega)$ is an isomorphism by the closed graph theorem. The bilinear form $(\mathbf{e}, \tilde{\mathbf{e}}) \in \mathbf{E}_1(\Omega) \times \mathbf{E}_1(\Omega) \to \int_\Omega \mathbf{A}\mathbf{e} : \tilde{\mathbf{e}}\, dx \in \mathbb{R}$ and the linear form $L \circ \mathcal{F}_1 : \mathbf{E}_1(\Omega) \to \mathbb{R}$ thus satisfy all the assumptions of the Lax–Milgram lemma. □

Note that, as shown in [2], the *Korn inequality in the space* $\mathbf{H}^1(\Omega)$ can be recovered as a simple corollary to Theorem 5.1, which thus provides a *new proof* of this classical inequality.

Consider likewise the *pure displacement problem of three-dimensional linearized elasticity*, which classically consists in finding $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ such that

$$J(\mathbf{u}) = \inf_{\mathbf{v} \in \mathbf{H}_0^1(\Omega)} J(\mathbf{v}), \quad \text{where } J(\mathbf{v}) = \frac{1}{2} \int_\Omega \mathbf{A} \nabla_s \mathbf{v} : \nabla_s \mathbf{v}\, \mathrm{d}x - L(\mathbf{v}),$$

where again $L(\mathbf{v}) = \int_\Omega \mathbf{f} \cdot \mathbf{v}\, \mathrm{d}x$ for some $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$. Thanks to Theorem 2.5, this problem can be again recast as *another quadratic minimization problem*, this time with $\boldsymbol{\varepsilon} := \nabla_s \mathbf{u} \in \mathbb{L}_s^2(\Omega)$ as the primary unknown:

Theorem 3.2. *Let Ω be a domain in $\mathbb{R}^3$. Define the Hilbert space*

$$\mathbf{E}_2(\Omega) := \left\{ \mathbf{e} \in \mathbb{L}_s^2(\Omega);\ \int_\Omega \mathbf{e} : \mathbf{s}\, \mathrm{d}x = 0 \text{ for all } \mathbf{s} \in \mathbb{L}_s^2(\Omega) \text{ satisfying } \mathbf{div}\, \mathbf{s} = \mathbf{0} \text{ in } \mathbf{H}^{-1}(\Omega) \right\},$$

and, for each $\mathbf{e} \in \mathbf{E}_2(\Omega)$, let $\mathcal{F}_2(\mathbf{e})$ denote the unique element in the space $\mathbf{H}_0^1(\Omega)$ that satisfies $\nabla_s \mathcal{F}_2(\mathbf{e}) = \mathbf{e}$ (Theorem 2.5). Then the mapping $\mathcal{F}_2 : \mathbf{E}_2(\Omega) \to \mathbf{H}_0^1(\Omega)$ defined in this fashion is an isomorphism between the Hilbert spaces $\mathbf{E}_2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$. Furthermore, the minimization problem: Find $\boldsymbol{\varepsilon} \in \mathbf{E}_2(\Omega)$ such that

$$j_2(\boldsymbol{\varepsilon}) = \inf_{\mathbf{e} \in \mathbf{E}_2(\Omega)} j_2(\mathbf{e}), \quad \text{where } j_2(\mathbf{e}) := \frac{1}{2} \int_\Omega \mathbf{A}\mathbf{e} : \mathbf{e}\, \mathrm{d}x - L \circ \mathcal{F}_2(\mathbf{e}),$$

has one and only one solution $\boldsymbol{\varepsilon}$, and this solution satisfies $\boldsymbol{\varepsilon} = \nabla_s \mathbf{u}$, where $\mathbf{u}$ is the unique minimizer of the functional J in the space $\mathbf{H}_0^1(\Omega)$.

Note that each one of the three minimization problems found above can be in turn immediately recast as yet another one, this time with the *linearized stress tensor* $\mathbf{A}\boldsymbol{\varepsilon}$ as the primary unknown, since the elasticity tensor field $\mathbf{A}$ is invertible almost everywhere in Ω.

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