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C. R. Acad. Sci. Paris, Ser. I 342 (2006) 923–926
COMPTES RENDUS MATHEMATIQUE
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Partial Differential Equations

Anisotropic harmonic maps into homogeneous manifolds: a compactness result

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Received 7 February 2006; accepted 14 March 2006

Available online 15 May 2006

Presented by Pierre-Louis Lions

Abstract

We introduce a new energy functional for maps between two manifolds, the critical points of which ($\tilde{p}$-harmonic maps) are solutions of a system of anisotropic quasilinear elliptic equations. In the case when the target is a homogeneous manifold with left invariant metric, we establish a compactness result for the corresponding $\tilde{p}$-harmonic maps. The proof relies on some deep results from harmonic analysis involving Hardy spaces. ***To cite this article: M. Sango, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Applications harmoniques anisotropes dans des variétés homogènes : un résultat de compacité. Nous introduisons une nouvelle fonctionnelle d'énergie pour des applications sur des variétés ; les points critiques de cette fonctionnelle (applications $\tilde{p}$-harmoniques) sont solutions d'un système d'équations elliptique, quasilinéaire, anisotrope. Dans le cas où la variété cible est homogène et munie d'une métrique invariante à gauche, nous établissons un résultat de compacité pour les applications $\tilde{p}$-harmoniques correspondantes. La démonstration utilise un résultat fondamental d'analyse harmonique dans des espaces de Hardy. ***Pour citer cet article : M. Sango, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Let M be a smooth open set bounded in $\mathbf{R}^m$ and N a n-dimensional compact smooth Riemannian manifold with the metric $g = (g_{ij})_{i,j=1,\dots,n}$. Let $\tilde{p} = (p_1, \dots, p_m) \in \mathbf{R}^m$, with $p_\alpha \geqslant 1$. For a C^1-map $f : M \to N$, we introduce the anisotropic $\tilde{p}$-energy:

$$E(f) = \int_M \sum_{\alpha=1}^{m} \frac{1}{p_\alpha} \left(g_{ij}\big(f(x)\big) \frac{\partial f^i}{\partial x^\alpha} \frac{\partial f^j}{\partial x^\alpha} \right)^{p_\alpha/2} \mathrm{d}x; \tag{1}$$

(here and in the sequel we omit the summation symbol over the indices i and j) the critical points of which satisfy the corresponding Euler–Lagrange equations,

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2006.03.018

$$\sum_{\alpha=1}^{m} \frac{\partial}{\partial x^\alpha}\left(|d_\alpha f|^{p_\alpha-2}\frac{\partial f^l}{\partial x^\alpha}\right) = -\sum_{\alpha=1}^{m} |d_\alpha f|^{p_\alpha-2}\Gamma_{ij}^l \frac{\partial f^i}{\partial x^\alpha}\frac{\partial f^j}{\partial x^\alpha}, \tag{2}$$

$l = 1, \ldots, n$, Γ_{ij}^l denotes the Christoffel symbols relative to the manifold N. This is a system of degenerate anisotropic quasilinear elliptic equations. The left-hand side of (2) will be denoted $\Delta_{\tilde p}$ and referred to as the $\tilde p$-Laplacian. We note that when all $p_\alpha = p$, $E(f)$ coincides with a p-energy whose critical points are p-harmonic maps. In the case when all $p_\alpha = 2$, we have the well-known energy for harmonic maps. These last cases have been considered by several authors starting from the pioneering works of Eells, Morrey and Sampson; for historical overview and extensive references, see the monographic masterpiece by Helein [5].

We now introduce some anisotropic Sobolev spaces. We define $W^1_{\tilde p}(M, \mathbf{R}^k)$ (with $p_\alpha \geqslant 1$, $\alpha = 1, \ldots, m$) as the space of functions $u : M \to \mathbf{R}^k$, $u(x) = (u_1(x), \ldots, u_k(x))$, such that each $u_i \in W^1_{\tilde p}(M)$;

$$W^1_{\tilde p}(M) = \left\{ v \in W^1_1(M)\colon\ \frac{\partial v}{\partial x^\alpha} \in L_{p_\alpha}(M),\ \alpha = 1, \ldots, m \right\},$$

$$\|v\|_{W^1_{\tilde p}(M)} = \|v\|_{L_1(M)} + \sum_{\alpha=1}^{m} \left\| \frac{\partial v}{\partial x^\alpha} \right\|_{L_{p_\alpha}(M)}.$$

Let N be isometrically embedded into $\mathbf{R}^k$, then $W^1_{\tilde p}(M, N)$ is the set of functions $u \in W^1_{\tilde p}(M, \mathbf{R}^k)$ such that $u(x) \in N$ for almost every $x \in M$.

Under appropriate geometric conditions on M (for instance M satisfies the so called weak l-horn condition ([1], §8–10) we have the following embedding theorem for anisotropic Sobolev spaces proved for instance in [7].

Theorem 1. *Set*:

$$\bar p^{-1} = \frac{\sum_{\alpha=1}^m p_\alpha^{-1}}{m} \quad \textit{and} \quad p^* = \frac{m\bar p}{m - \bar p} \quad \textit{if } \bar p < m. \tag{3}$$

If $\bar p < m$, then

$$W^1_{\tilde p}(M) \hookrightarrow L_q(M) \tag{4}$$

compactly for each $q \in (1, \max\{p^*, p_i\})$.

Definition 2. A function $f \in W^1_{\tilde p}(M, N)$ is a weakly $\tilde p$-harmonic map of M into N provided the equations (2) hold in the sense of distributions.

In this Note we shall be concerned with the compactness properties of solutions of (2) when the target N is a homogeneous manifold with left-invariant metric. The corresponding problem for p-harmonic maps was established by Luckhaus [8] and extended to homogeneous target case by Toro and Wang [9]; we refer also to [6] for the evolution case.

Our approach is inspired from [9] with a strong harmonic analysis flavor centered around some deep results from Hardy spaces and the analog of a celebrated result by Coifman, Lions, Meyer and Semmes [2] that we derive for anisotropic Sobolev spaces which are the natural energy spaces for $\tilde p$-harmonic maps. We note that the present work is the first in which $\tilde p$-harmonic maps are being considered. It seems also to be the first where anisotropic Sobolev spaces which constitute a very important class of function spaces are being applied to geometric variational problems.

The main result of the Note is the following:

Theorem 3. *Let $p_\alpha \geqslant 2$, $\alpha = 1, \ldots, m$. Let M be such that Theorem* 1 *holds and assume that (N, g) is a compact homogeneous space with a left invariant metric g. Let $\{u_k\}_{k=1,2,\ldots}$ be a sequence of weakly $\tilde p$-harmonic maps in $W^1_{\tilde p}(M, N)$ which converges weakly to u in $W^1_{\tilde p}(M, N)$. Then $u : M \to N$ is a weakly $\tilde p$-harmonic map.*

2. Auxiliary results

Owing to the celebrated Nash embedding theorem, N can be isometrically embedded into some Euclidean space $\mathbf{R}^k$, in view of the compactness of N. Let $i : N \to \mathbf{R}^k$ be the embedding. Then the function $F = i \circ f$ with values in $\mathbf{R}^k$ satisfies the orthogonality condition $\Delta_{\tilde{p}} F \perp T_F N$, where $\Delta_{\tilde{p}}$ is the $\tilde{p}$-Laplacian with respect to M and $\mathbf{R}^k$, if f satisfies (2).

Let X be a Killing vector field on N. That is the generator of an isometry of N, satisfying $\langle D_v X(p), v\rangle = 0$, at all $p \in N$ and for all $v \in T_p N$. Here $\langle\cdot,\cdot\rangle$ stands for the inner product in $\mathbf{R}^k$ restricted to $T_p N$.

Let f satisfy (2). Then by Noether's Theorem ([4] or [5]), the tangent vector field,

$$|d_\alpha f|^{p_\alpha - 2}\left\langle X(f), \frac{\partial f}{\partial x^\alpha}\right\rangle,$$

is divergence free in the distributional sense. In other words for any $\xi \in C_o^\infty(M)$,

$$\sum_{\alpha=1}^{m} \int_M \left\langle \frac{\partial}{\partial x^\alpha}\big(\xi X(f)\big), |d_\alpha f|^{p_\alpha - 2}\frac{\partial f}{\partial x^\alpha}\right\rangle \mathrm{d}x = 0. \tag{5}$$

Differentiating in (5) and using the Killing property of X, we get:

$$\sum_{\alpha=1}^{m} \int_M \frac{\partial}{\partial x^\alpha}\left\langle X(f), |d_\alpha f|^{p_\alpha - 2}\frac{\partial f}{\partial x^\alpha}\right\rangle \mathrm{d}x = 0. \tag{6}$$

A result of Helein [4] stipulates that on a homogeneous space N (represented as the quotient $N = G/H$ of a Lie group G by its closed subgroup H) of dimension n with a left invariant metric, there exist l smooth tangent vector fields $Y_1, \ldots, Y_l$ and l Killing fields $X_1, \ldots, X_l$ on N such that any vector $V \in T_y N$ $(y \in N)$ admits the expansion $V = \sum_{i=1}^{l}\langle X_i, V\rangle Y_i$; l is the dimension of the Lie algebra $\mathcal{G}$ of G. From this expansion and the divergence freeness of $|d_\alpha f|^{p_\alpha - 2}\langle \frac{\partial f}{\partial x^\alpha}, X_i\rangle$, it follows that

$$\sum_{\alpha=1}^{m} \frac{\partial}{\partial x^\alpha}\left(|d_\alpha f|^{p_\alpha - 2}\frac{\partial f}{\partial x^\alpha}\right) = \sum_{\alpha=1}^{m}\sum_{i=1}^{l} |d_\alpha f|^{p_\alpha - 2}\left\langle \frac{\partial f}{\partial x^\alpha}, X_i(f)\right\rangle \frac{\partial Y_i(f)}{\partial x^\alpha}, \tag{7}$$

weakly in M. This system of equations is equivalent to (2).

We establish the following generalization of Coifman, Lions, Meyer and Semmes's result [2] who proved it when all the $p_\alpha = p$.

Proposition 4. *Let $f \in W^1_{\tilde{p}}(M)$ and let $E = (E_1, \ldots, E_m)$ be a vector function such that each component $E_\alpha \in L_{p'_\alpha}(M)$, $((p'_\alpha)^{-1} + p_\alpha^{-1} = 1,\ p_\alpha > 1)$, $\alpha = 1, \ldots, m$. Suppose that $\operatorname{div} E = 0$ in the weak sense. Then $\langle \nabla f, E\rangle \in \mathcal{H}_{1,\mathrm{loc}}(M)$ and for any compact set $K \subset M$, there exists a constant $C > 0$, such that*

$$\big\|\langle \nabla f, E\rangle\big\|_{\mathcal{H}_1(K)} \leqslant C \sum_{\alpha=1}^{m}\left[\|E_\alpha\|^{p'_\alpha}_{L_{p'_\alpha}(M)} + \left\|\frac{\partial f}{\partial x^\alpha}\right\|^{p_\alpha}_{L_{p_\alpha}(M)}\right], \tag{8}$$

here ∇f is the gradient of f and $\mathcal{H}_1$ denotes Hardy's space.

3. Proof of Theorem 3

Let the sequence $\{f_k\}_{k=1,2,\ldots} \in W^1_{\tilde{p}}(M, N)$ satisfy the system of Eqs. (7), i.e.,

$$\sum_{\alpha=1}^{m} \frac{\partial}{\partial x^\alpha}\left(|d_\alpha f_k|^{p_\alpha - 2}\frac{\partial f_k}{\partial x^\alpha}\right) = g_k \tag{9}$$

weakly, where

$$g_k =: \sum_{\alpha=1}^{m}\sum_{i=1}^{l} |d_\alpha f_k|^{p_\alpha - 2}\left\langle \frac{\partial f_k}{\partial x^\alpha}, X_i(f_k)\right\rangle \frac{\partial Y_i(f_k)}{\partial x^\alpha}.$$

We have:

$$f_k \rightharpoonup f \quad \text{weakly in } W^1_{\tilde{p}}(M,N). \tag{10}$$

Thus $\{f_k\}$ is uniformly bounded in $W^1_{\tilde{p}}(M,N)$. Hence each component $\{f^i_k\}$ is uniformly bounded in $W^1_{\tilde{p}}(M)$. In view of Theorem 1, it follows that

$$f^i_k \to f^i \quad \text{strongly in } L_q(M),\ \text{with } q \in \big(1, \max\{p_\alpha, p^*\}\big).$$

Therefore

$$f^i_k \to f^i \quad \text{a.e., in } M. \tag{11}$$

Also (10) implies that

$$|d_\alpha f_k|^{p_\alpha - 2}\frac{\partial f_k}{\partial x^\alpha} \rightharpoonup |d_\alpha f|^{p_\alpha - 2}\frac{\partial f}{\partial x^\alpha} \quad \text{weakly in } L_{p'_\alpha}(M,N). \tag{12}$$

Arguing as in ([3], pp. 409–411) modulo some straightforward adaptations, we get that the function

$$\theta_k = \sum_{\alpha=1}^{m}\left(|d_\alpha f_k|^{p_\alpha - 2}\frac{\partial f_k}{\partial x^\alpha} - |d_\alpha f|^{p_\alpha - 2}\frac{\partial f}{\partial x^\alpha}\right)\frac{\partial (f_k - f)}{\partial x^\alpha}$$

converges to zero almost everywhere in M. Thus

$$\frac{\partial f_k}{\partial x^\alpha} \to \frac{\partial f}{\partial x^\alpha} \quad \text{a.e. in } M. \tag{13}$$

Since X_i and Y_i are smooth vectors fields, it follows from (11) and (13) that

$$g_k \to g \quad \text{a.e. in } M, \tag{14}$$

and $g \in L_1(M)$. Further by Theorem 4, we also have that $g_k \in \mathcal{H}_{1,\mathrm{loc}}(M)$ and for a compact set $K \subset M$

$$\|g_k\|_{\mathcal{H}_1(K)} \leqslant C\sum_{\alpha=1}^{m}\left\|\frac{\partial f_k}{\partial x^\alpha}\right\|^{p_\alpha}_{L_{p_\alpha}(M,\mathbf{R}^m)}. \tag{15}$$

All the ingredients are now in place for the proof of the identity,

$$\sum_{\alpha=1}^{m}\int_M |d_\alpha f|^{p_\alpha - 2}\frac{\partial f}{\partial x^\alpha}\frac{\partial \varphi}{\partial x^\alpha}\,\mathrm{d}x = \int_M g\varphi\,\mathrm{d}x, \tag{16}$$

for any $\varphi \in W^1_{\tilde{p}}(M,\mathbf{R}^k) \cap L_\infty(M,\mathbf{R}^k)$ with support in the compact set $K \subset M$. (16) which concludes the proof the theorem, follows from the relations (10)–(15) together with arguments along the lines of [9].

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