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Complex Analysis

A new approach to Cullen-regular functions of a quaternionic variable

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Abstract

In this Note we announce the basic elements and results of a new theory of regular functions of one quaternionic variable. The theory we describe follows an idea of Cullen, but we use a more geometric approach to show that it is possible to build a rather complete theory. *To cite this article: G. Gentili, D.C. Struppa, C. R. Acad. Sci. Paris, Ser. I 342 (2006).* © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé

Une nouvelle approche aux fonctions Cullen-régulières d'une variable quaternionelle. Dans ce travail nous annonçons les éléments et les résultats de base d'une nouvelle théorie des fonctions régulières d'une variable quaternionelle. La théorie que nous décrivons ici s'inspire d'une idée de Cullen, mais nous utilisons une approche plus géométrique pour montrer qu'il est possible de construire une théorie plutôt complète. *Pour citer cet article : G. Gentili, D.C. Struppa, C. R. Acad. Sci. Paris, Ser. I 342 (2006).* © 2006 Académie des sciences. Published by Elsevier SAS. All rights reserved.

1. Introduction

Let \mathbb{H} be the skew field of real quaternions. Its elements are of the form $q = x_0 + ix_1 + jx_2 + kx_3$ where the x_l are real, and i, j, k are imaginary units (i.e. their square equals -1) such that ij = -ji = k, jk = -kj = i, and ki = -ik = j. Since the beginning of last century, mathematicians have been interested in creating a theory of quaternionic valued functions of a quaternionic variable, which would somehow resemble the classical theory of holomorphic functions of one complex variable.

Interesting theories have been introduced. The best known is due to Fueter, [6], who defined the differential operator

$$\frac{\partial}{\partial \bar{q}} = \frac{1}{4} \left(\frac{\partial}{\partial x_0} + i \frac{\partial}{\partial x_1} + j \frac{\partial}{\partial x_2} + k \frac{\partial}{\partial x_3} \right)$$

now known as the Cauchy–Fueter operator, and defined the space of regular functions as the space of solutions of the equation associated to this operator. The theory of regular functions is by now very well developed, in many different

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directions, and we refer the reader to [12] for the basic features of these functions. More recent work in this area can also be found in [3]. While the theory is extremely successful in replicating many important properties of holomorphic functions (and not only in one variable), the major disappointment is that even the identity f(q) = q, and therefore polynomials and series in q, fail to be regular in the sense of Fueter.

A new definition was given by Cullen in [4] on the basis of the notion of intrinsic functions as developed in [11]. This definition has the advantage that polynomials, and even power series of the form $\sum_{n=0}^{\infty} q^n a_n$, $a_n \in \mathbb{H}$, are regular in this sense.

Polynomials of a quaternionic variable, and power series, are also included in the interesting class of holomorphic functions over quaternions, which was introduced by Fueter [5] and more recently generalized and developed by Laville and Ramadanoff [8,9], who built the theory of holomorphic Cliffordian functions. If Δ denotes the Laplacian, the (left) holomorphic functions over quaternions are the solutions of the equation associated to the differential operator $\frac{\partial}{\partial q}\Delta$. It turns out that the set of Cullen regular functions and the set of Fueter regular functions, strictly contained in the set of holomorphic functions over quaternions, do not coincide.

Cullen regular functions are also closely related to a class of functions of the reduced quaternionic variable $x_0 + ix_1 + jx_2$, studied by H. Leutwiler [10]. This class consists of all the solutions of a generalized Cauchy–Riemann system of equations, it contains the natural polynomials and supports the series expansion of its elements as well.

Let us denote by S the unit sphere of purely imaginary quaternions, i.e. $S = \{q = ix_1 + jx_2 + kx_3 \in \mathbb{H} \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}$. Notice that if $I \in S$, then $I^2 = -1$; for this reason the elements of S are called imaginary units. We can rephrase in a more geometrical term the definition of Cullen [4].

Definition 1.1. Let Ω be a domain in \mathbb{H} . A real differentiable function $f: \Omega \to \mathbb{H}$ is said to be *C*-regular if, for every $I \in \mathbb{S}$, its restriction f_I to the complex line $L_I = \mathbb{R} + \mathbb{R}I$ passing through the origin and containing 1 and *I* is holomorphic on $\Omega \cap L_I$.

Throughout the Note, since no confusion can arise, we will refer to C-regular functions as regular functions tout court.

Remark 1.2. The requirement that $f : \Omega \to \mathbb{H}$ is regular is equivalent to require that, for every I in S,

$$\bar{\partial}_I f(x+yI) := \frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f_I(x+yI) = 0$$

on $\Omega \cap L_I$.

Remark 1.3. The Cauchy–Fueter differential operator defines a derivative in the classical Fréchet sense; on the other hand, the definition of regularity which we have just provided can be interpreted in the spirit of the Gateaux derivative.

Still in the spirit of Gateaux, we can define a notion of *I*-derivative as follows:

Definition 1.4. Let Ω be a domain in \mathbb{H} and let $f : \Omega \to \mathbb{H}$ be a real differentiable function. For any $I \in \mathbb{S}$ and any point $q = x + yI \in \Omega$ (*x* and *y* real numbers) we define the *I*-derivative of *f* in *q* by

$$\partial_I f(x+yI) := \frac{1}{2} \left(\frac{\partial}{\partial x} - I \frac{\partial}{\partial y} \right) f_I(x+yI)$$

In the next section we will show that when f is regular we can use this definition to introduce a notion of derivative (which we will call Cullen derivative).

In this Note we announce several results which show that, on the basis of this definition, it is possible to construct a significant and interesting theory for regular functions. The detailed proofs of our results will appear elsewhere [7].

2. Power series and series expansions for regular functions

In order to study polynomials and power series in q, we first note that the basic polynomial $q^n a$, with a a quaternion, is regular according to Definition 1.1. Since the sum of regular functions is regular, we immediately have that

polynomials with quaternionic coefficients on the right are regular. In order to consider power series $\sum_{n=0}^{\infty} q^n a_n$, we will endow the space of regular functions with the natural uniform convergence on compact sects. The same arguments which hold for complex power series, see e.g. [1], allow us to obtain the analog of the Abel's theorem. Since convergence of power series is uniform on compact sets, Abel's theorem implies, in particular, that power series are regular in their domain of convergence.

As we indicated earlier, we can introduce a notion of derivative for regular functions:

Definition 2.1. Let Ω be a domain in \mathbb{H} , and let $f : \Omega \to \mathbb{H}$ be a regular function. The Cullen derivative of f, $\partial_C f$, is defined as follows:

$$\partial_C(f)(q) = \begin{cases} \partial_I(f)(q) & \text{if } q = x + yI \text{ with } y \neq 0, \\ \frac{\partial f}{\partial x}(x) & \text{if } q = x \text{ is real.} \end{cases}$$

This definition of derivative is well posed because it is applied only to regular functions. In fact, the value of the derivative at a real point x can be computed using different imaginary units, and a priori there is no reason why the values which one obtains should coincide. However, if a function f is regular, its derivative in the point x is immediately shown to be equal to $\frac{\partial f}{\partial x}(x)$. It is easy to construct examples which show the necessity of the regularity of f. Note that this phenomenon is peculiar of the quaternionic case, and does not appear in the complex case. The reason for this is that the unit sphere of imaginary numbers has dimension 2 in the case of quaternions, but it is only made of two points, $\{i, -i\}$, in the complex case.

Let f be a regular function. Since for every I in S it is $\bar{\partial}_I(\partial_I(f)) = \partial_I(\bar{\partial}_I(f)) = 0$, we obtain that the Cullen derivative of a regular function is still regular.

Note also that the derivative of a power series can be done term by term because of the uniform convergence, so that

$$\partial_C\left(\sum_{n=0}^{\infty}q^n a_n\right) = \sum_{n=1}^{\infty}nq^{n-1}a_n.$$

This new series has the same radius of convergence of the original series.

In what follows, we will always restrict our attention to functions which are regular on a ball B = B(0, R) centered in the origin and of radius R. The following lemma is simple to prove but is essential for all the results in this Note:

Lemma 2.2. If f is a regular function on B, then for every $I \in S$, and every J in S, perpendicular to I, there are two holomorphic functions $F, G: B \cap L_I \to L_I$ such that for any z = x + yI, it is

$$f_I(z) = F(z) + G(z)J.$$

It is now possible to deduce, [7]:

Theorem 2.3. Any regular function $f : B \to \mathbb{H}$ has a series expansion of the form

$$f(q) = \sum_{n=0}^{\infty} q^n \frac{1}{n!} \frac{\partial^n f}{\partial x^n}(0),$$

and is therefore analytic.

The two previous results are instrumental in proving the analog, for regular functions, of many well known results from the theory of holomorphic functions such as the identity principle, the maximum modulus, the Cauchy representation, the Liouville and the Morera theorem. Though the results are what one might expect, the proofs rely upon the characterization given in Lemma 2.2 and are given in detail in [7].

The following refinement of Lemma 2.2 is of some independent interest as it helps understand the geometry of these regular functions.

Lemma 2.4. Under the same hypotheses of Lemma 2.2, either $F \equiv 0$ on every L_I , or there is at most one imaginary unit I such that $F \equiv 0$ on L_I . Similarly, either $G \equiv 0$ on every L_J , or there is at most one imaginary unit J such that $G \equiv 0$ on L_J . Moreover, if there exists an imaginary unit I such that $F \equiv 0$ on L_I (resp. $G \equiv 0$ on L_I), then there is no other unit J such that $G \equiv 0$ on L_J (resp. $F \equiv 0$ on L_J), unless $f \equiv 0$.

3. The Schwarz lemma, the unit ball and the right half space

The proof of the classical Schwarz Lemma is based on the maximum modulus principle for holomorphic functions, and on the fact that any holomorphic function defined on the open unit ball of \mathbb{C} has a series expansion centered at $0 \in \mathbb{C}$. Since suitably modified versions of these results hold for any regular function on the open unit ball *B* of \mathbb{H} , the Schwarz Lemma can be proved in its natural formulation for regular functions.

Composition of regular functions is not regular in general, and hence the set of all biregular transformations of the open unit ball *B* of \mathbb{H} is not a group under composition. Nevertheless, the quaternionic right half space defined as $\mathbb{H}^+ = \{q = x_0 + ix_1 + jx_2 + kx_3 \in \mathbb{H}: x_0 > 0\}$ turns out to be diffeomorphic to the open unit ball *B* via the biregular Cayley transformation defined by $C(q) = (1 - q)^{-1}(1 + q) = 1 + 2(\sum_{n>1} q^n)$.

On the basis of all these results, it is clear that one can develop an interesting theory for C-regular functions, also in the more general environment of Clifford Algebras (see, e.g. [2]). Further developments will be the object of a subsequent paper.

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