Extended Picard complexes for algebraic groups and homogeneous spaces

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Abstract

For a smooth geometrically integral algebraic variety $X$ over a field $k$ of characteristic $0$, we define the extended Picard complex $\text{UPic}(\overline{X})$. It is a complex of length 2 which combines the Picard group $\text{Pic}(\overline{X})$ and the group $U(\overline{X}) := \overline{k}[\overline{X}] / \overline{k}^\times$, where $\overline{k}$ is a fixed algebraic closure of $k$ and $\overline{X} = X \times_k \overline{k}$. For a connected linear $k$-group $G$ we compute the complex $\text{UPic}(\overline{G})$ (up to a quasi-isomorphism) in terms of the algebraic fundamental group $\pi_1(\overline{G})$. We obtain similar results for a homogeneous space $X$ of a connected $k$-group $G$. To cite this article: M. Borovoi, J. van Hamel, C. R. Acad. Sci. Paris, Ser. I 342 (2006).

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Résumé

Complexes de Picard étendus pour des groupes algébriques et des espaces homogènes. Soient $k$ un corps de caractéristique zéro et $X$ une $k$-variété algébrique lisse et géométriquement intégrée. Nous définissons le complexe de Picard étendu $\text{UPic}(\overline{X})$. C’est un complexe de longueur 2 qui combine le groupe de Picard $\text{Pic}(\overline{X})$ et le groupe $U(\overline{X}) := \overline{k}[\overline{X}] / \overline{k}^\times$, où $\overline{k}$ est une clôture algébrique fixée de $k$ et $\overline{X} = X \times_k \overline{k}$. Pour un $k$-groupe linéaire connexe $G$, nous calculons le complexe $\text{UPic}(\overline{G})$ (à quasi-isomorphisme près) en termes du groupe fondamental algébrique $\pi_1(\overline{G})$. Nous obtenons des résultats similaires pour un espace homogène $X$ d’un $k$-groupe connexe $G$. Pour citer cet article : M. Borovoi, J. van Hamel, C. R. Acad. Sci. Paris, Ser. I 342 (2006).

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Throughout the Note, $k$ denotes a field of characteristic 0 and $\overline{k}$ is a fixed algebraic closure of $k$. By a $k$-group we mean a linear algebraic group defined over $k$.

Let $G$ be a connected reductive $k$-group. Let

$$\rho : G^{\text{sc}} \to G^{\text{ss}} \hookrightarrow G$$

be Deligne’s homomorphism, where $G^{\text{ss}}$ is the derived subgroup of $G$ (it is semisimple) and $G^{\text{sc}}$ is the universal covering of $G^{\text{ss}}$ (it is simply connected). Let $T \subset G$ be a maximal torus (defined over $k$) and let $T^{\text{sc}} := \rho^{-1}(T)$ be the
corresponding maximal torus of $G^{\text{sc}}$. The 2-term complex of tori

$$T^{\text{sc}} \xrightarrow{\rho} T$$

(with $T^{\text{sc}}$ in degree $-1$) plays an important role in the study of the arithmetic of reductive groups. For example, the Galois hypercohomology $H^1(k, T^{\text{sc}} \rightarrow T)$ of this complex is the abelian Galois cohomology of $G$ (cf. [1]). The corresponding Galois module

$$X_*(T)/\rho_*X_*(T^{\text{sc}})$$

(where $X_*$ denotes the cocharacter group of a torus) is called the algebraic fundamental group $\pi_1(\overline{G})$ (loc. cit.). The related complex group with holomorphic $\text{Gal}(\overline{k}/k)$-action

$$\text{Hom}(\pi_1(\overline{G}), C^\times) = \ker(X^*(T) \otimes C^\times \rightarrow X^*(T^{\text{sc}}) \otimes C^\times)$$

(where $X^*$ denotes the character group of an algebraic group) is canonically isomorphic to the center of a connected Langlands dual group $\overline{G}$ for $G$, considered by Kottwitz [7].

Clearly, the above constructions rely on the linear algebraic group structure of $\overline{G}$. However we show in this note that they are related to a very natural geometric/cohomological construction that works for an arbitrary smooth $k$-variety $X$. The proofs will be published elsewhere.

1. The extended Picard complex

By a $k$-variety we mean a smooth geometrically integral $k$-variety. If $X$ is a $k$-variety, we write $\overline{X}$ for $X \times_k \overline{k}$. We write $k[X]$ (resp. $\overline{k}(X)$) for the ring of regular functions (resp. the field of rational functions) on $\overline{X}$.

For a $k$-variety $X$, consider the cone $\text{UPic}(\overline{X})$ of the morphism

$$G_m(\overline{k}) \rightarrow \tau_{\leq 1}R\Gamma(\overline{X}, G_m)$$

in the derived category of discrete Galois modules. More explicitly, this cone is represented by the 2-term complex

$$\overline{k}(X)^\times/\overline{k}^\times \rightarrow \text{Div}(\overline{X})$$

(with $\overline{k}(X)^\times/\overline{k}^\times$ in degree 0), where Div denotes the divisor group. It follows from the definitions that the cohomology groups $H^i$ of the complex $\text{UPic}(\overline{X})$ vanish for $i \neq 0, 1$, and

$$H^0(\text{UPic}(\overline{X})) = U(\overline{X}) := \overline{k}[\overline{X}^\times/\overline{k}^\times], \quad H^1(\text{UPic}(\overline{X})) = \text{Pic}(\overline{X}).$$

Hence $\text{UPic}(\overline{X})$ can be regarded as a 2-extension of $\text{Pic}(\overline{X})$ by $U(\overline{X})$. We shall call this complex the extended Picard complex of $X$.

**Lemma 1.1.** Let $X_c$ be a smooth compactification of a $k$-variety $X$. Then there is a distinguished triangle

$$\text{UPic}(\overline{X}) \rightarrow \text{Div}_{\overline{X}_c \setminus \overline{X}}(\overline{X}) \rightarrow \text{Pic}(\overline{X}_c) \rightarrow \text{UPic}(\overline{X})[1]$$

where $\text{Div}_{\overline{X}_c \setminus \overline{X}}(\overline{X})$ is the permutation module of divisors in the complement of $\overline{X}$ in $\overline{X}_c$.

Now we consider $\text{Pic}(X) = H^1(X, G_m)$ and $\text{Br}(X) = H^2_{\text{et}}(X, G_m)$ (over $k$). Consider the canonical homomorphisms $\text{Br}(k) \xrightarrow{\alpha} \text{Br}(X) \xrightarrow{\beta} \text{Br}(\overline{X})$ and set $\text{Br}_a(X) = \ker \beta/\text{im} \alpha$.

**Lemma 1.2.** Let $X$ be a $k$-variety.

(i) There is a natural injection $\text{Pic}(X) \hookrightarrow H^1(k, \text{UPic}(\overline{X}))$, which is an isomorphism if $X(k) \neq \emptyset$.

(ii) There is a natural injection $\text{Br}_a(X) \hookrightarrow H^2(k, \text{UPic}(\overline{X}))$, which is an isomorphism if $X(k) \neq \emptyset$ or if $H^3(k, G_m) = 0$ (e.g. when $k$ is a number field).

If $C$ is a complex of $\text{Gal}(\overline{k}/k)$-modules, we write $\prod_{\omega} H^1(k, C) = \ker[H^1(k, C) \rightarrow \bigoplus \gamma H^1(\gamma, C)]$ where $\gamma$ runs over all closed procyclic subgroups of $\text{Gal}(\overline{k}/k)$.
Proposition 1.3. Let $X_c$ be a smooth compactification of a smooth $k$-variety $X$. The triangle of Lemma 1.1 gives rise to an isomorphism

$$\Pi^1_{\text{et}}(k, \text{Pic}(\overline{X}_c)) \sim \Pi^2_{\text{et}}(k, \text{UPic}(\overline{X})).$$

This is particularly interesting for a homogeneous variety $X$ of a connected $k$-group $G$ with connected geometric stabilizer, for which we have $\Pi^1_{\text{et}}(k, \text{Pic}(\overline{X}_c)) = H^1(k, \text{Pic}(\overline{X}_c))$, see [4].

2. Algebraic groups and torsors

Let $G$ be a connected reductive $k$-group. We define the dual complex $\pi_1(G)^D$ to $\pi_1(G)$ by

$$\pi_1(G)^D = \left( \mathbb{X}^*(\overline{T}) \to \mathbb{X}^*(\overline{T}^{sc}) \right) \quad \text{(with } \mathbb{X}^*(\overline{T}) \text{ in degree 0).}$$

Theorem 2.1. For a connected reductive $k$-group $G$ there is a canonical, functorial in $G$ isomorphism (in the derived category of discrete Galois modules)

$$\text{UPic}(G) \sim \pi_1(G)^D.$$

Let $G$ be any connected linear $k$-group, not necessarily reductive. We write $G^u$ for the unipotent radical of $G$, and set $G^{\text{red}} = G/G^u$ (it is reductive). We define $\pi_1(G) := \pi_1(G^{\text{red}})$. 

Corollary 2.2. For any connected linear $k$-group $G$ we have a canonical isomorphism $\text{UPic}(G) \sim \pi_1(G)^D$.

Combining Corollary 2.2 with Lemma 1.2, we find a new proof of the following result.

Corollary 2.3 (Kottwitz [7]). For any connected linear $k$-group $G$ we have canonical isomorphisms $\text{Pic}(G) \sim H^1(k, \pi_1(G)^D)$ and $\text{Br}_2(G) \sim H^2(k, \pi_1(G)^D)$.

Theorem 2.1 gives a description of the complex UPic for a $k$-torsor as well, thanks to the following result which is a straightforward generalization of [8, Lemme 6.7]).

Proposition 2.4. Let $G$ be a connected linear $k$-group and let $X$ be a $k$-torsor under $G$. There is a canonical isomorphism $\text{UPic}(X) \sim \text{UPic}(G)$, functorial in $G$ and $X$, in the derived category of discrete Galois modules.

Combining the fact that $\Pi^1_{\text{et}}(k, \text{Pic}(\overline{X}_c)) = H^1(k, \text{Pic}(\overline{X}_c))$ for any smooth compactification $\overline{X}_c$ of a $k$-torsor $X$ under $G$ (cf. [3]) with Proposition 1.3, Proposition 2.4, and Corollary 2.2, we obtain a new proof of the following result.

Corollary 2.5 (Borovoi–Kunyavski˘ı [2]). With $G$ and $X$ as above, $H^1(k, \text{Pic}(\overline{X}_c)) \simeq \Pi^2_{\text{et}}(k, \pi_1(G)^D)$.

3. Homogeneous spaces

Let $G$ be a connected $k$-group such that $\text{Pic}(\overline{G}) = 0$ (i.e. $(G^{\text{red}})^{\text{ss}}$ is simply connected). Let $X$ be a homogeneous space of $G$ defined over $k$. Let $\bar{x} \in X(\bar{k})$, and let $\overline{H}$ be the stabilizer of $\bar{x}$ in $\overline{G}$. Then $\text{Gal}(\bar{k}/k)$ acts on $\mathbb{X}^*(\overline{H})$. We do not assume that $X$ has a $k$-point or that $\overline{H}$ is connected.

Theorem 3.1. For $G$ and $X$ as above, there is an isomorphism

$$\text{UPic}(\overline{X}) \sim \left( \mathbb{X}^*(\overline{G}) \to \mathbb{X}^*(\overline{H}) \right) \quad \text{(with } \mathbb{X}^*(\overline{G}) \text{ in degree 0)}$$

in the derived category of discrete Galois modules. In particular, there is an exact sequence

$$0 \to U(\overline{X}) \to \mathbb{X}^*(\overline{G}) \to \mathbb{X}^*(\overline{H}) \to \text{Pic}(\overline{X}) \to 0.$$
The exact sequence of Theorem 3.1 generalizes an exact sequence of Fossum–Iversen [6, Proposition 3.1] and Sansuc [8, Proposition 6.10]. Note that the requirement $\text{Pic}(\overline{G}) = 0$ is not a serious restriction, since for any connected $k$-group $G$ we can find a surjective homomorphism $G' \rightarrow G$ with $\text{Pic}(\overline{G'}) = 0$.

**Corollary 3.2.** For $G$ and $X$ as above there are injections $\text{Pic}(X) \hookrightarrow H^1(k, \mathbf{X}^*(\overline{G}) \rightarrow \mathbf{X}^*(\overline{H}))$ and $\text{Br}_a(X) \hookrightarrow H^2(k, \mathbf{X}^*(\overline{G}) \rightarrow \mathbf{X}^*(\overline{H}))$, which are isomorphisms if $X(k) \neq \emptyset$.

The corollary follows from Theorem 3.1 and Lemma 1.2.

**4. The elementary obstruction**

Let $X$ be a $k$-variety. We have an extension of complexes of Galois modules

$$0 \rightarrow \mathbf{k}^\times \rightarrow (\bar{k}(\mathbf{X})^\times \rightarrow \text{Div}(\mathbf{X})) \rightarrow (\bar{k}(\mathbf{X})^\times / \mathbf{k}^\times \rightarrow \text{Div}(\mathbf{X})) \rightarrow 0.$$ 

It defines an element $e(X) \in \text{Ext}^1(\text{UPic}(\mathbf{X}), \mathbf{k}^\times)$. If $X$ has a $k$-point, then this extension splits (in the derived category), hence $e(X) = 0$. By slight abuse of terminology we call this class $e(X)$ the **elementary obstruction** to the existence of a $k$-point in $X$ (cf. [5, Définition 2.2.1 and Proposition 2.2.4]).

When $X$ is a $k$-torsor under a $k$-group $G$, Proposition 2.4 and Theorem 2.1 give us that $\text{UPic}(\mathbf{X}) = \pi_1(\overline{G})^D$. We obtain

$$\text{Ext}^1(\text{UPic}(\mathbf{X}), \mathbf{k}^\times) = H^1(k, \text{Hom}(\pi_1(\overline{G})^D, \mathbf{k}^\times)) = H^1(k, \mathbf{X}_*(\mathbf{T}^{sc}) \otimes \mathbf{k}^\times \rightarrow \mathbf{X}_*(\mathbf{T}) \otimes \mathbf{k}^\times) = H^1(k, \mathbf{T}^{sc} \rightarrow \mathbf{T})$$

(where $\mathbf{T}^{sc}$ is in degree $-1$). The abelian group $H^1(k, G) := H^1(k, \mathbf{T}^{sc} \rightarrow \mathbf{T})$ is called the first abelian Galois cohomology group of $G$, and in [1] an abelianization map $\text{ab}^1: H^1(k, G) \rightarrow H^1_{ab}(k, G)$ was constructed. Here we compute the elementary obstruction $e(X) \in H^1_{ab}(k, G)$ in terms of the cohomology class $\text{cl}(X) \in H^1(k, G)$.

**Theorem 4.1.** Let $X$ be a $k$-torsor under a connected $k$-group $G$. With the above notation we have $e(X) = \text{ab}^1(\text{cl}(X))$ (up to sign).

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**References**