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Partial Differential Equations

Convexity of solutions of parabolic equations

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Abstract

We establish here necessary and sufficient conditions for the propagation of convexity in parabolic equations. We consider as well linear equations and fully nonlinear ones. And we discuss several variants and extensions of these results. ***To cite this article: P.-L. Lions, M. Musiela, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Convexité des solutions d'équations paraboliques. Nous établissons dans cette note des conditions nécessaires et suffisantes pour la propagation de la convexité des solutions d'équations paraboliques. Nous considérons aussi bien des équations linéaires que complètement non linéaires. Et nous mentionnons diverses variantes et extensions de ces résultats. ***Pour citer cet article : P.-L. Lions, M. Musiela, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***
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Version française abrégée

Nous obtenons et prouvons dans cette note des conditions nécessaires et suffisantes pour la propagation de la convexité des solutions d'équations elliptiques du second ordre, éventuellement dégénérées aussi bien dans le cas linéaire que dans le cas complètement non linéaire. Plus précisément, nous considérons la solution (de viscosité, voir [2]) de

$$\frac{\partial u}{\partial t} - a_{ij}\partial_{ij}u = 0, \quad x \in \mathbf{R}^N,\ t > 0 \tag{1}$$

ou

$$\frac{\partial u}{\partial t} + F\big(x, D_x^2 u\big) = 0, \quad x \in \mathbf{R}^N,\ t > 0 \tag{2}$$

vérifiant la condition initiale

$$u|_{t=0} = u_0(x) \quad \text{sur } \mathbf{R}^N. \tag{3}$$

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Les conditions précises sur a et F sont données dans la version anglaise ci-dessous.

Nous dirons que la convexité est préservée pour l'Éq. (1) (ou l'Éq. (2)) si u est convexe en x pour tout $t \geqslant 0$ chaque fois que u_0 est convexe.

Nos principaux résultats – dont les démonstrations sont esquissées dans la version anglaise ci-dessous – sont les suivants :

Théorème 2.1. *La convexité est préservée pour* (1) *si et seulement si la condition suivante est satisfaite*

$$\big(a_{ij}(x+y)\xi_i\xi_j\big)^{1/2} \text{ est convexe en } y \in (\mathbf{R}\xi)^{\perp},\ \forall x \in \mathbf{R}^N,\ \forall \xi \neq 0 \in \mathbf{R}^N.$$

Ce résultat permet bien sûr de retrouver les cas bien connus où la convexité est préservée ($N = 1$, a_{ij} quadratique ...). Il implique également que si $N \geqslant 2$ et si a est sous quadratique à l'infini (par exemple bornée !) alors la convexité n'est préservée que dans le cas trivial où a est constante.

Théorème 3.1. *La convexité est préservée pour* (2) *si et seulement si la condition suivante est satisfaite*

$$F(x+y, B) \text{ est concave en } y \in (\mathbf{R}\xi)^{\perp},\ A \in X_\xi,\ \forall x \in \mathbf{R}^N,\ \forall \xi \neq 0 \in \mathbf{R}^N$$
$$\text{où } X_\xi = \big\{A\, N \times N \text{ symétrique},\ A\xi = 0,\ A > 0 \text{ sur } (\mathbf{R}\xi)^{\perp}\big\},\ B = A^{-1} \text{ sur } (\mathbf{R}\xi)^{\perp} \text{ et } B\xi = 0.$$

Enfin, nous considérons dans la section 4 ci-dessous diverses variantes et extensions de ces résultats.

1. Introduction

We obtain and prove in this note necessary and sufficient conditions for the convexity of solutions of general second order elliptic, possibly degenerate, equations both in the linear case and in the fully nonlinear one. More precisely, we consider uniformly continuous (in $x \in \mathbf{R}^N$, $t \geqslant 0$ bounded) solutions u, say in viscosity sense – see the User's guide [2] by M.G. Grandall, H. Ishii and P.-L. Lions –, of

$$\frac{\partial u}{\partial t} - a_{ij}\partial_{ij}u = 0, \quad x \in \mathbf{R}^N,\ t > 0 \tag{1}$$

or

$$\frac{\partial u}{\partial t} + F\big(x, D_x^2 u\big) = 0, \quad x \in \mathbf{R}^N,\ t > 0, \tag{2}$$

with the initial condition

$$u|_{t=0} = u_0(x) \quad \text{in } \mathbf{R}^N, \tag{3}$$

where u_0 is **convex** and (for instance) uniformly continuous on $\mathbf{R}^N$. Here and everywhere below, $a = \frac{1}{2}\sigma \cdot \sigma^{\mathrm{T}}$ where σ is Lipschitz (for instance) from $\mathbf{R}^N$ into the space of $N \times m$ matrices, $N, m \geqslant 1$; u is real valued; $D^2 u = D_x^2 u = (\partial_{ij}u)_{1 \leqslant i,j \leqslant N}$ stands for the Hessian matrix and $F = F(x, A)$ is continuous over $\mathbf{R}^N \times S^N$ (the space of $N \times N$ symmetric matrices), nonincreasing with respect to A and satisfies the classical uniqueness structure condition for viscosity solutions (see [2]).

We shall say that **convexity is preserved** for (1) (or (2)) if u is convex in x for all $t \geqslant 0$ whenever u_0 is convex. In Section 2 below, we establish a necessary and sufficient condition on a (or σ) for preserving convexity in the case of Eq. (1). We then derive in Section 3 an analogous result in the fully nonlinear case (2). Finally, we briefly mention in Section 4 several extensions and variants.

The question solved in this note is both a natural and classical one and we refer the reader to N.J. Korevaar [7]; A.U. Kennington [6]; B. Kawohl [5]; H. Ishii and P.-L. Lions [4]; Y. Giga, S. Goto, H. Ishii and M.H. Sato [3]. In these works (some of which address closely related issues), sufficient conditions are derived by the use of the maximum principle (basically for 'D^2u'). More recently, a more geometrical approach, through the use of the convex envelope of u, was introduced by O. Alvarez, J.-M. Lasry and P.-L. Lions [1] allowing to obtain rather general sufficient conditions. The fact that we obtain necessary and sufficient conditions appears to be new. Our methods of proof (in fact, we present below two different approaches for the 'sufficient part', one of which is an extension of the geometrical argument introduced in [1]) are also somewhat different or more general than the ones used in the above references, although, as is to be expected, we do use the maximum principle.

Our main motivation for studying this question and related ones (mentioned in Section 4) stems from mathematical modelling in Finance. Indeed, the fact that convexity is preserved is an important and natural requirement for a financial model (see, for instance, M. Romano and N. Touzi [8]). Finally, let us mention (and this is clearly related to the application to Mathematical Finance) the probabilistic interpretation of our results in the linear case. Let X_t be the diffusion process corresponding to a i.e. denoting by W_t a standard n-dimensional Brownian motion

$$\mathrm{d}X_t = \sigma(X_t)\cdot \mathrm{d}W_t, \quad X_0 = x \in \mathbf{R}^N. \tag{4}$$

Then, the solution of (1) corresponding to the initial condition (3) is given by $u(x,t) = E[u_0(X_t)]$ and the question about preserving convexity is then equivalent to the question of the convexity in x (for all $t > 0$) of $E[u_0(X_t)]$ whenever u_0 is convex. Let us make two observations at this stage: (i) the fact that convexity is preserved if $N = 1$ for (1) is well-known and easy to check by analytical or probabilistic arguments..., (ii) we do not know whether the main result in section 1 below can be proved by a purely probabilistic argument.

2. The linear case

Theorem 2.1. *Convexity is preserved for* (1) *if and only if the following condition holds*

$$\left|\sigma^{\mathrm{T}}(x+y)\cdot\xi\right| = 2\big(a(x+y)\cdot\xi,\xi\big)^{1/2} \text{ is convex in } y\in(\mathbf{R}\xi)^{\perp}, \text{ for all } x\in\mathbf{R}^N, \text{ for all } \xi\neq 0\in\mathbf{R}^N. \tag{5}$$

Remarks and examples. (i) Obviously, (5) holds if $N = 1$ or if σ is affine in x.

(ii) If a is strictly subquadratic at infinity ($\frac{a(x)}{|x|^2}\to 0$ as $|x|\to\infty$), then, when $N \geqslant 2$, (5) holds if and only if a is constant!

(iii) If $N = m$, $\sigma_{ij} = \varphi_i(x)\delta_{ij}$ where $\varphi_i \geqslant 0$ is convex in x, then (5) holds.

(iv) The condition (5) is, as it should be (!), invariant by linear transformations of $\mathbf{R}^N$. It is also satisfied for λa and $a_1 + a_2$ if $\lambda > 0$ and (5) holds for a, a_1, a_2. This is to be expected in view of simple arguments involving rescaling time (for λa) and Trotter formula (for $a_1 + a_2$).

(v) If a is smooth (say C^2), (5) is equivalent to: $\forall|\xi| = 1, \forall|\eta| = 1,\ \xi\cdot\eta = 0$

$$\left(\frac{\partial^2 a_{ij}}{\partial\eta^2}(x)\xi_i\xi_j\right)\big(a_{ij}(x)\xi_i\xi_j\big) \geqslant \frac{1}{2}\left(\frac{\partial a_{ij}}{\partial\eta}\xi_i\xi_j\right)^2 \quad \text{on } \mathbf{R}^N. \tag{6}$$

Sketch of proof of Theorem 2.1. We begin with a few preliminary observations. First of all, it is enough to prove the result when a is uniformly elliptic (i.e. $a(x) \geqslant \nu I_N$ on $\mathbf{R}^N$ for some $\nu > 0$) and smooth. This follows easily from a simple approximation argument on one hand and from Remark (iv) by regularization and addition of νI_N to a on the other hand.

At this stage, the strategy of proof is the following. In order to prove that (5) is a sufficient condition, we consider a solution u of (1) for an arbitrary smooth convex initial condition u_0 such that $\alpha I_N \leqslant D_x^2 u_0 \leqslant \beta I_N$ on $\mathbf{R}^N$ (for some $0 < \alpha < \beta < \infty$). And we look at $t_0 = \inf\{t \geqslant 0 \mid \inf[\frac{\partial^2 u}{\partial\xi^2}(x,t) \mid x\in\mathbf{R}^N,\ |\xi| = 1] = 0\}$ and we assume without loss of generality that $0 < t_0 < \infty$. By appropriate perturbation arguments which are somewhat classical in the use of maximum principle, we may assume without loss of generality that there exist $x_0\in\mathbf{R}^N,\ |\xi| = 1$ such that $\frac{\partial^2 u}{\partial\xi^2}(x_0,t_0) = 0$ and it is then enough to show that we have

$$\frac{\partial}{\partial t}\left\{\frac{\partial^2 u}{\partial\xi^2}(x_0,t_0)\right\} \geqslant 0. \tag{7}$$

In order to do so, we observe that $w = \frac{\partial^2 u}{\partial\xi^2}$ satisfies

$$\frac{\partial w}{\partial t} - a_{ij}\partial_{ij}w - 2\frac{\partial a_{ij}}{\partial\xi}\partial_{ij}\frac{\partial u}{\partial\xi} - \frac{\partial^2 a_{ij}}{\partial\xi^2}\partial_{ij}u = 0 \tag{8}$$

and that we have by the convexity of u at time $t = t_0$

$$D^2u(x_0,t_0) \geqslant 0, \frac{\partial^2 u}{\partial \xi^2}(x_0,t_0) = 0, \quad \nabla \frac{\partial^2 u}{\partial \xi^2}(x_0,t_0) = 0, \quad \frac{\partial^4 u}{\partial \xi^4}(x_0,t_0) \geqslant 0. \tag{9}$$

The method of proof then consists in expanding u at x_0 to the fourth order, deduce the restrictions on this 4th order polynomial from convexity and prove (7) using (8) and the condition (5). Up to a rotation and a translation, we may assume without loss of generality that $x_0 = 0, \xi = e_N$ and write $x = (y,z) \in \mathbf{R}^{N-1} \times \mathbf{R}$.

We thus write (using (9))

$$\begin{aligned} u(\cdot,t_0) = {} & c_0 + c_1 \cdot y + c_2 z + \frac{1}{2}\big\{A_{ij} y_i y_j + B_{ij} y_i y_j z + D_{ij} y_i y_j z^2 \\ & + \alpha z^4 + \beta_i y_i z^3 + \gamma_{ijk} y_i y_j y_k z + \varepsilon_{ijk} y_i y_j y_k + \delta_{ijkl} y_i y_j y_k y_l + \circ\big(|y|^2 + z^2\big)^2\big\}, \end{aligned} \tag{10}$$

for some $c_0, c_2, \alpha \in \mathbf{R}$; $\beta, c_1 \in \mathbf{R}^{N-1}$; symmetric matrices A, B and D and tensors γ, ε and δ. Using (9) and (10), (8) reduces to

$$\frac{\partial w}{\partial t} = \frac{\partial^2 a_{ij}}{\partial z^2} A_{ij} + 2\frac{\partial a_{ij}}{\partial z} B_{ij} + 2a_{ij} D_{ij} + 12\alpha a_{NN} + 6\beta_i a_{iN}. \tag{11}$$

Next, we write the fact that $D^2u(\cdot,t_0) \geqslant 0$ and we observe that without loss of generality, we may take $\gamma = \varepsilon = \delta = 0$ since they do not appear in (11) nor do they matter in the inequality below hence obtaining for all $\zeta \in \mathbf{R}^{d-1}$, $\Theta \in \mathbf{R}$

$$\begin{aligned} & A_{ij}\zeta_i\zeta_j + B_{ij}\zeta_i\zeta_j z + D_{ij}\zeta_i\zeta_j z^2 + 2B_{ij} y_i \zeta_j \Theta + 4D_{ij} y_i \zeta_j z \Theta \\ & \quad + 6\alpha z^2 \Theta^2 + (D_{ij} y_i y_j)\Theta^2 + 3\beta_i \zeta_i z^2 \Theta + 3\beta_i y_i z \Theta^2 \geqslant 0, \end{aligned} \tag{12}$$

for z small. Moreover, we deduce that we have for all $Y \in \mathbf{R}^{N-1}$, $\zeta \in \mathbf{R}^{N-1}$, $\Theta \in \mathbf{R}$

$$A_{ij}\zeta_i\zeta_j + 2B_{ij} Y_i \zeta_j + D_{ij} Y_i Y_j + 6\alpha\Theta^2 + 3\beta_i \zeta_i \Theta \geqslant 0. \tag{13}$$

Thus there only remains to show that (13) and (5) imply that the right-hand side of (11) is nonnegative.

We then write (13) as follows

$$A_{ij}\zeta_i\zeta_j + 2B_{ij} Y_i \zeta_j + \left(D_{ij} - \frac{3}{8\alpha}\beta_i\beta_j\right) Y_i Y_j + 6\alpha\left(\Theta + \frac{1}{4\alpha}\beta_i Y_i\right)^2 \geqslant 0 \tag{14}$$

(assuming without loss of generality that $\alpha > 0$). Next, we observe that

$$2a_{ij}\left(\frac{3}{8\alpha}\beta_i\beta_j\right) + 12\alpha a_{NN} + 6\beta_i a_{iN} = 12\alpha\left\{a_{NN} + 2a_{iN}\left(\frac{\beta_i}{4\alpha}\right) + a_{ij}\frac{\beta_i}{4\alpha}\frac{\beta_j}{4\alpha}\right\} \geqslant 0, \quad \text{since } a_{ij} \geqslant 0,$$

and we thus only need to consider the case when $\alpha = \beta = 0$.

And there only remains to prove that if A, B, D are symmetric matrices (we may assume that $A, D > 0$ without loss of generality) satisfying

$$A_{ij}\zeta_i\zeta_j + 2B_{ij} Y_i \zeta_j + D_{ij} Y_i Y_j > 0 \tag{15}$$

and $K = \frac{\partial^2 a_{ij}}{\partial z^2}$, $L = \frac{\partial a_{ij}}{\partial z}$, $M = a_{ij}$ satisfy in view of (6)

$$(K_{ij}\eta_i\eta_j)(M_{ij}\eta_i\eta_j) \geqslant \frac{1}{2}(L_{ij}\eta_i\eta_j)^2, \tag{16}$$

then we have

$$\mathrm{Tr}(KA) + 2\,\mathrm{Tr}(LB) + 2\,\mathrm{Tr}(MD) \geqslant 0. \tag{17}$$

This is an elementary exercize in matrix analysis which can be checked easily once we remark that (15) is equivalent to $D \geqslant BA^{-1}B$, and that (16) is equivalent to

$$(K\zeta,\zeta) + 2\lambda(L\zeta,\zeta) + 2\lambda^2(M\zeta,\zeta) \geqslant 0, \quad \forall \zeta \in \mathbf{R}^{N-1},\ \forall \lambda \in \mathbf{R}.$$

Then, replacing for instance ζ by $A^{1/2}\zeta$, K by $A^{1/2}KA^{1/2}$, L by $A^{1/2}LA^{1/2}$, M by $A^{1/2}MA^{1/2}$ and diagonalizing $A^{-1/2}BA^{-1/2}$, we deduce the following inequality which yields (17)

$$\operatorname{Tr}(KA) + 2\operatorname{Tr}(LB) + 2\operatorname{Tr}(MBA^{-1}B) \geqslant 0. \tag{18}$$

The above proof allows also to prove the fact that (5) is a necessary condition: indeed, if (6) does not hold for some $x \in \mathbf{R}^N$, $|\eta| = 1$ that we may always assume, without loss of generality, to be $x = 0$, $\eta = e_N$, the above argument shows clearly that there exist symmetric $(N-1)\times(N-1)$ matrices A, B, D such that $A > 0$, $D > 0$, $D > BA^{-1}B$ and

$$\frac{\partial^2 a_{ij}}{\partial z^2}A_{ij} + 2\frac{\partial a_{ij}}{\partial z}B_{ij} + 2a_{ij}D_{ij} < 0. \tag{19}$$

We then choose u_0 to be $\frac{1}{2}\{A_{ij}y_iy_j + B_{ij}y_iy_jz + D_{ij}y_iy_jz^2\}$ on a neighborhood of 0 on which this polynomial is convex (and we extend it to be a convex function on $\mathbf{R}^N$ by standard extension arguments). And we conclude since (19) implies that we have

$$\frac{\partial}{\partial t}\left\{\frac{\partial^2 u}{\partial z^2}\right\}(0,0) < 0 \quad \text{while } \frac{\partial^2 u}{\partial z^2}(0,0) = \frac{\partial^2 u_0}{\partial z^2}(0,0) = 0. \qquad \square$$

3. The fully nonlinear case

Theorem 3.1. *Convexity is preserved for* (2) *if and only if the following condition holds for all* $\xi \neq 0 \in \mathbf{R}^N$ *and for all* $x \in \mathbf{R}^N$

$$\begin{aligned}&F(x+y, B) \text{ is concave in } y \in (\mathbf{R}\xi)^{\perp},\ A \in X_\xi,\\ &\text{where } X_\xi = \{A \in S^N,\ A\xi = 0,\ A > 0 \text{ on } (\mathbf{R}\xi)^{\perp}\},\ B = A^{-1} \text{ on } (\mathbf{R}\xi)^{\perp} \text{ and } B\xi = 0.\end{aligned} \tag{20}$$

Remarks. (i) The above result and its proof is an extension of the main result in [1], where it is shown that convexity is preserved for (2) if $F(x, A^{-1})$ is concave in $(x, A) \in \mathbf{R}^N \times S^N$, $A > 0$.

(ii) If $F(x, B) = -\operatorname{Tr}(a_{ij}(x)B_{ij})$ (and (2) reduces to (1)), one can show that (20) is equivalent to (5) (as it should!).

Sketch of proof of Theorem 3.1. The fact that (20) is necessary is shown exactly as in the proof of Theorem 2.1 (after a regularization and approximation argument that we do not detail here).

In order to prove that (20) is also a sufficient condition, we follow and extend the proof introduced in [1] which consists in showing that the convex envelope $v = u^{**}$ of u is a (viscosity) supersolution on (2). Hence, by the comparison principle for viscosity solutions, $u^{**} \geqslant u$ and thus $u^{**} = u$.

Next, we explain the modification in the argument of [1] in the case when u is smooth. Moreover, we only need to consider points $(x_0, t_0) \in \mathbf{R}^N \times (0, \infty)$ such that $v(x_0, t_0) < u(x_0, t_0)$. Then there exist

$$k \in \{2, \ldots, N+1\}; \quad x_1, \ldots, x_k \in \mathbf{R}^N; \quad \lambda_1, \ldots, \lambda_k \in (0,1)$$

such that

$$\sum_{i=1}^{k}\lambda_i = 1, \quad \sum_{i=1}^{k}\lambda_i x_i = x_0, \quad \sum_{i=1}^{k}\lambda_i u(x_i) = v(x_0) \quad \text{and} \quad x_2 - x_1, \ldots, x_k - x_1$$

are linearly independent. We assume that $k = 2$, the other cases then follow by a single geometrical method arguing inductively on k (notice for instance that if $k = 3$, $N = 2$, v is affine near $x_0 \ldots$). We then denote by $\xi = x_2 - x_1$ and we observe that v is affine on the segment $[x_1, x_2]$. We then argue formally as if v were twice differentiable at (x_0, t_0) (the justification in general following as in [1] from the study of the superjet of v). We then remark that we have $\frac{\partial^2 v}{\partial \xi^2}(x_0, t_0) = 0$. On the other hand, see [1] for more details, we have

$$\begin{aligned}&\nabla_x v(x_0,t_0) = \nabla_x u(x_1,t_0) = \nabla_x u(x_2,t_0),\\ &\frac{\partial v}{\partial t}(x_0,t_0) \geqslant \frac{\partial u}{\partial t}(x_1,t_0), \qquad \frac{\partial v}{\partial t}(x_0,t_0) \geqslant \frac{\partial u}{\partial t}(x_2,t_0),\\ &A \leqslant \left(\lambda_1 A_1^{-1} + \lambda_2 A_2^{-1}\right)^{-1} \quad \text{on } (\mathbf{R}\xi)^{\perp}, \quad A_1 \geqslant 0, \quad A_2 \geqslant 0\end{aligned} \tag{21}$$

where $A = D_x^2 v(x_0, t_0)$, $A_1 = D_x^2 u(x_1, t_0)$, $A_2 = D_x^2 u(x_2, t_0)$.

Hence, we have using (21) and the 'ellipticity' of F

$$\frac{\partial v}{\partial t}(x_0,t_0)+F(x_0,A)\geqslant \lambda_1\frac{\partial u}{\partial t}(x_1,t_0)+\lambda_2\frac{\partial u}{\partial t}(x_2,t_0)+F(\lambda_1x_1+\lambda_2x_2,B)$$

where $B=(\lambda_1A_1^{-1}+\lambda_2A_2^{-1})^{-1}$ on $(\mathbf{R}\xi)^\perp$ and $B\xi=0$. We then deduce from (20) the following inequality

$$\frac{\partial v}{\partial t}(x_0,t_0)+F(x_0,A)\geqslant \lambda_1\left(\frac{\partial u}{\partial t}(x_1,t_0)+F\big(x_1,\tilde A_1\big)\right)+\lambda_2\left(\frac{\partial u}{\partial t}(x_2,t_0)+F\big(x_2,\tilde A_2\big)\right)$$

where $\tilde A_i=A_i$ on $(\mathbf{R}\xi)^\perp$, $\tilde A_i\xi=0$, and thus $\tilde A_i\leqslant A_i$ $(i=1,2)$.

And we conclude using the 'ellipticity' of F

$$\frac{\partial v}{\partial t}(x_0,t_0)+F(x_0,A)\geqslant \lambda_1\left(\frac{\partial u}{\partial t}(x_1,t_0)+F(x_1,A_1)\right)+\lambda_2\left(\frac{\partial u}{\partial t}(x_2,t_0)+F(x_2,A_2)\right)=0. \qquad \square$$

Of course, the above argument can be used in the special case of a linear equation in which case (2) reduces to (1) and thus provides another proof of the fact that (5) is a sufficient condition in Theorem 2.1.

4. Extensions and variants

Many extensions and variants are possible. Let us mention without further detail equations involving t, u and D_xu (i.e. $F=F(x,t,u,D_xu,D_x^2u)$), obstacle problems, elliptic equations instead of parabolic equations or equations involving integro-differential terms (corresponding to jump diffusion processes). Boundary conditions are more delicate to handle except for 'state constraints' boundary conditions (see [1] for more detail on the role of boundary conditions…). Another variant, of interest for financial applications, consists in considering a class of initial conditions, say when $N=1$, u_0 of the following form $u_0(x)=(x-K)_+$ ('calls'). We may then study the **joint** convexity in (x,K) of the solution of (1) (or (2)).

Another extension which can be solved by the methods introduced here concerns the partial convexity of u with respect to a set of variables y whenever u_0 is itself convex in y. However, a much more delicate problem consists in asking the same question whenever u_0 is a convex function of y only. It turns out that this second question is relevant for financial applications and our preliminary results in that direction indicate that, in general, the solution is not always convex except for very particular examples (which have to be studied specifically).

Another direction that can be studied with our methods is the propagation of $C^{1,1}$ bounds or of semi-convexity bounds. In the case of the propagation of semi-convexity bounds, we consider $C_0\geqslant 0$ and ask whether the solution u of (2) satisfies for all $t\geqslant 0$ (for instance)

$$u(x,t)+\frac{1}{2}C_0|x|^2 \text{ is convex on } \mathbf{R}^N \tag{22}$$

whenever $u_0+\frac{1}{2}C_0|x|^2$ is convex on $\mathbf{R}^N$. Obviously, a necessary and sufficient condition is deduced from Theorem 3.1 considering $F(x,A-C_0I_N)$ in place of $F(x,A)$.

This observation allows to prove the following result

Theorem 4.1. *Let F be continuous, nonincreasing on S^N and assume that F satisfies for some $R>0$*

$$F(A) \text{ is concave in } A \text{ if } \lambda_1(A)\leqslant -R \tag{23}$$

where $\lambda_1(A)$ is the first eigenvalue of A. Let $u_0\in UC(\mathbf{R}^N)$ be semi-convex on $\mathbf{R}^N$. Then, the viscosity solution $u\in UC(\mathbf{R}^N\times[0,T])$ $(\forall T\in(0,\infty))$ satisfies (22) for all $t\in(0,\infty)$ for some $C_0\geqslant R$.

In particular, if F is uniformly elliptic i.e. if F satisfies for some $\nu>0$

$$F(A+B)\leqslant F(A)-\nu\operatorname{Tr}(B),\quad \forall A\in S^N,\ \forall B\in S^N,\ B\geqslant 0, \tag{24}$$

then $u\in L_t^\infty(C_x^{1,1})$ whenever $u_0\in C^{1,1}$ (i.e. $D_x^2u_0\in L_{t,x}^\infty$).

It suffices indeed to observe that (20) holds for $F(A - C_0 I_N)$ for $C_0 \geqslant R$ since $\lambda_1(B - C_0 I_N) \leqslant -C_0 \leqslant -R$ and $\lambda_1(A - C_0 I_N) \leqslant -C_0 \leqslant R$ (using the convexity properties of $A \to A^{-1}$ and the 'ellipticity' of F as in [1]).

The detailed proofs of the results presented in this Note and the various variants and extensions mentioned in this section will be detailed elsewhere.

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