Partial Differential Equations

Some asymptotic properties for solutions of one-dimensional advection–diffusion equations with Cauchy data in $L^p(\mathbb{R})$

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Abstract

We state and discuss a number of fundamental asymptotic properties of solutions $u(\cdot, t)$ to one-dimensional advection–diffusion equations of the form

$$u_t + f(u)_x = (a(u)u_x)_x,$$

$x \in \mathbb{R}$, $t > 0$, assuming initial values $u(\cdot, 0) = u_0 \in L^p(\mathbb{R})$ for some $1 \leq p < \infty$.


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Résumé

Quelques propriétés asymptotiques des solutions des équations d’avection–diffusion unidimensionnelles aux données initiales dans $L^p(\mathbb{R})$. Nous établissons plusieurs propriétés asymptotiques fondamentales des solutions $u(\cdot, t)$ des équations d’avection–diffusion du type $u_t + f(u)_x = (a(u)u_x)_x$, $x \in \mathbb{R}$, $t > 0$, aux données initiales dans l’espace de Lebesgue $L^p(\mathbb{R})$, où $1 \leq p < \infty$. Pour citer cet article : P. Braz e Silva, P.R. Zingano, C. R. Acad. Sci. Paris, Ser. I 342 (2006).

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1. Introduction

We examine some properties of solutions $u(\cdot, t)$ to the Cauchy problem for scalar advection–diffusion equations

$$u_t + f(u)_x = (a(u)u_x)_x,$$

$x \in \mathbb{R}$, $t > 0$, (1a)

$$u(\cdot, 0) = u_0 \in L^p(\mathbb{R}), \quad 1 \leq p < \infty,$$ (1b)

where $a(\cdot)$ and $f(\cdot)$ are given smooth functions. We assume that $a(u) \geq \mu > 0$ for some fixed constant $\mu$ and all values of $u$ concerned. Under these conditions, it is known that problem (1) admits a unique, smooth (classical), globally defined solution $u(\cdot, t) \in C^0([0, \infty[, L^p(\mathbb{R}))$, which is bounded for $t > 0$ and satisfies (1b) in $L^p$ sense, that is, $\|u(\cdot, t) - u_0\|_{L^p(\mathbb{R})} \to 0$ as $t \to 0$, see e.g. [2–4] and Section 2 below. Several additional properties of $u(\cdot, t)$ are given next.

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2. Decay estimates

We first state an important energy-type inequality for \( u(\cdot, t) \).

**Theorem 2.1.** Assume \( a(\cdot), f(\cdot) \in C^1 \) with \( a(\cdot) \) bounded below by some constant \( \mu > 0 \). Then, for each \( q \geq \max\{p, 2\} \), the solution \( u(\cdot, t) \in C^0([0, \infty[, L^p(\mathbb{R})] \) of the Cauchy problem (1) satisfies

\[
\begin{align*}
T^{\frac{q}{2p}} \|u(\cdot, T)\|_{L^p(\mathbb{R})}^{q} + q(q - 1)\mu \int_{0}^{T} \int_{\mathbb{R}} |u(x, t)|^{q - 2} |u_x(x, t)|^2 \, dx \, dt \\
\leq 2 \left( \frac{q}{2p} \right)^{\frac{1}{2} + 1} \left( 1 - \frac{1}{q} \right)^{-\frac{1}{2}} \mu^{\frac{1}{2}} \left( \frac{q}{r} - 1 \right)^{-\frac{1}{2}} \|u_0\|_{L^p(\mathbb{R})}^{q} T^{\frac{1}{2}}
\end{align*}
\]

(2)

for all \( T > 0 \).

Theorem 2.1 can be proved adapting the method discussed in [5] to our present needs. Decay estimates for \( u(\cdot, t) \) are readily obtained from inequality (2). Indeed, by the maximum principle and choosing \( q = 4p \), inequality (2) yields the supnorm estimate

\[
\|u(\cdot, t)\|_{L^\infty(\mathbb{R})} \leq C_p \|u_0\|_{L^p(\mathbb{R})} (\mu t)^{-\frac{1}{4}}
\]

(3)

for all \( t > 0 \). We note that the minimal value for \( C_p \) is not known; its particular value given here is not optimal. Now, since \( \|u(\cdot, t)\|_{L^p(\mathbb{R})} \leq \|u_0\|_{L^p(\mathbb{R})} \) for all \( t \geq 0 \), it follows from a simple interpolation estimate that

\[
\|u(\cdot, t)\|_{L^r(\mathbb{R})} \leq C_p \left( \frac{r}{p} \right)^{\frac{1}{2}} \|u_0\|_{L^p(\mathbb{R})} (\mu t)^{-\frac{1}{2} - \frac{1}{4}}
\]

(4)

for all \( p \leq r \leq \infty \). Therefore, solutions decay in \( L^r \) for any \( r > p \). Using standard estimates for fundamental solutions of linear parabolic problems (see [1,2]), one obtains decay rates for the derivatives of \( u(\cdot, t) \) as well, e.g.

\[
\|u_x(\cdot, t)\|_{L^p(\mathbb{R})} \leq C(p, r, \mu, K_p, t_0) \|u_0\|_{L^p(\mathbb{R})} t^{\frac{1}{2} - \frac{1}{2} - \frac{1}{4}}
\]

(5)

for each \( t_0 > 0 \). Similar bounds hold for the other derivatives. Here, the constant \( C(p, r, \mu, K_p, t_0) \) depends on the particular functions \( a(\cdot) \) and \( f(\cdot) \), on the values of \( p, r, \mu, t_0 \), and on \( K_p \), a bound for \( \|u_0\|_{L^p(\mathbb{R})} \), i.e., \( K_p > 0 \) chosen such that

\[
\|u_0\|_{L^p(\mathbb{R})} \leq K_p.
\]

(6)

3. Asymptotic behavior, \( p = 1 \)

For \( t \gg 1 \), more detailed behavior of \( u(\cdot, t) \) can be obtained from Theorem 3.1 below. This theorem follows from the estimates given in Section 2. Here, we assume \( a, f \in C^2 \), with \( f \) Hölder continuous at 0. We also assume \( a(u) \geq \mu > 0 \) for all \( u \), as before.

**Theorem 3.1.** Let \( u_0 \in L^1(\mathbb{R}) \), and let \( v(\cdot, t) \in C^0([0, \infty[, L^1(\mathbb{R})] \) be (any) solution of the Burgers equation

\[
v_t + f'(0)v_x + f''(0)v^2 = a(0)v_{xx}, \quad x \in \mathbb{R}, \ t > 0
\]

(7)

having the same mass as \( u(\cdot, t) \), i.e., \( \int_{\mathbb{R}} v(x, 0) \, dx = \int_{\mathbb{R}} u_0(x) \, dx \). Then, one has

\[
\lim_{t \to \infty} t^{\frac{1}{2}} (1 - \frac{1}{r}) \|u(\cdot, t) - v(\cdot, t)\|_{L^r(\mathbb{R})} = 0
\]

(8)

for each \( 1 \leq r \leq \infty \), uniformly in \( r \).

Now, solutions of (7) can be studied in detail through explicit representation formulas obtained with the so-called Hopf–Cole transformation, see [6]. In many cases, these properties can be recast for (1a) using Theorem 3.1 above, as illustrated by the following two results.
Theorem 3.2. Let \( u(\cdot, t), \hat{u}(\cdot, t) \in C^0([0, \infty), L^1(\mathbb{R})) \) be solutions of Eq. (1a) corresponding to initial states \( u_0, \hat{u}_0 \in L^1(\mathbb{R}) \), respectively, with the same mass. Then, one has
\[
\lim_{t \to \infty} t^{\frac{1}{2}(1 - \frac{1}{p})} \| u(\cdot, t) - \hat{u}(\cdot, t) \|_{L^r(\mathbb{R})} = 0,
\]
for all \( 1 \leq r \leq \infty \), uniformly in \( r \).

Theorem 3.3. Let \( u(\cdot, t) \in C^0([0, \infty), L^1(\mathbb{R})) \) be the solution of problem (1) for some given initial state \( u_0 \in L^1(\mathbb{R}) \) with mass \( m \). Then, for each \( 1 \leq r \leq \infty \),
\[
\lim_{t \to \infty} t^{\frac{1}{2}(1 - \frac{1}{p})} \| u(\cdot, t) \|_{L^r(\mathbb{R})} = \gamma_r(m),
\]
where, if \( f''(0) \neq 0 \), the quantity \( \gamma_r(m) \) is given by
\[
\gamma_r(m) = \frac{|m|}{\sqrt{4\pi a(0)}} \left( \frac{2a(0)}{mf''(0)} \right)^{\frac{1}{2}} \left( 1 - e^{-\frac{mf''(0)}{4a(0)}} \right) \| F \|_{L^r(\mathbb{R})},
\]
for \( F \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \) defined by
\[
F(x) = 2e^{-x^2}/\left( 1 + e^{-\frac{mf''(0)}{4a(0)}} - \left( 1 - e^{-\frac{mf''(0)}{4a(0)}} \right) \text{erf}(x) \right).
\]
Here, \( \text{erf}(x) \) is the error function, and \( \gamma_r(0) = 0 \). If \( f''(0) = 0 \), \( \gamma_r(m) \) is given by
\[
\gamma_r(m) = \frac{|m|}{\sqrt{4\pi a(0)}} \left( \frac{4\pi a(0)}{r} \right)^{\frac{1}{2}}.
\]

The case \( r = 1 \) is worth explicit mention:
\[
\lim_{t \to \infty} \| u(\cdot, t) \|_{L^1(\mathbb{R})} = |m| \quad \text{and} \quad \lim_{t \to \infty} \| u(\cdot, t) - \hat{u}(\cdot, t) \|_{L^1(\mathbb{R})} = 0
\]
for any two solutions \( u(\cdot, t), \hat{u}(\cdot, t) \) of Eq. (1a) transporting the same mass \( m \).

4. Asymptotic behavior, \( p > 1 \)

Lastly, we consider solutions \( u(\cdot, t) \) of problem (1) when \( p > 1 \). In this case, taking cut-off approximations \( v_R = u_0 \cdot \chi_{[-R,R]} \in L^1(\mathbb{R}) \cap L^p(\mathbb{R}) \) of the given initial data \( u_0 \in L^p(\mathbb{R}) \), and using results for the case \( p = 1 \), one obtains the following.

Theorem 4.1. Let \( u(\cdot, t) \in C^0([0, \infty), L^p(\mathbb{R})) \) be the solution of problem (1) corresponding to an initial state \( u_0 \in L^p(\mathbb{R}) \), \( p > 1 \). Then, one has
\[
\lim_{t \to \infty} t^{\frac{1}{2}(\frac{1}{p} - \frac{1}{r})} \| u(\cdot, t) \|_{L^r(\mathbb{R})} = 0
\]
for each \( 1 \leq p \leq r \leq \infty \), uniformly in \( r \).

As for the heat equation, the convergence to zero in (10) can be arbitrarily slow (for suitable \( u_0 \in L^p(\mathbb{R}) \) verifying (6), \( K_p > 0 \) fixed). Therefore, no rates better than (4) can be given in general.

References