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Differential Topology

Self-coincidence of mappings between spheres and the Strong Kervaire Invariant One Problem

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Abstract

Let $f: S^{4n-2} \to S^{2n}$ be a map between spheres of dimensions $4n-2$ and $2n$ with $n > 4$. We show that the existence of such a map satisfying the property that the pair $(f, f): S^{4n-2} \to S^{2n}$ can be deformed to a coincidence free pair but cannot be deformed to coincidence free by small deformation is equivalent to the Strong Kervaire Invariant One Problem, i.e., the existence of an element of order 2 with Kervaire invariant one in the stable homotopy group π^s_{2n-2}. ***To cite this article: D. Gonçalves, D. Randall, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Les auto-coincidences d'applications entre deux sphères et la forme forte du problème de Kervaire. Soit $f: S^{4n-2} \to S^{2n}$ une application continue entre les sphères de dimensions respectives $4n-2$ et $2n$ pour $n > 4$. Nous démontrons que, si la paire (f, f) est déformable en une paire libre de coïncidences, alors elle n'est pas déformable par petites déformations si et seulement si $n = 2^j$, $j \geqslant 3$, et l'invariant de Kervaire de la classe d'homotopie $[f] \in \pi_{4n-2}(S^{2n})$ est 1. Cette dernière condition est équivalente à une forme forte du problème de Kervaire. ***Pour citer cet article : D. Gonçalves, D. Randall, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

The study of self-coincidences of mappings between spheres in [5] and [7] investigated the relation between homotopy disjointness and homotopy disjointness by small deformation for a map $f: S^m \to S^{2n}$. The pair (f, f) is homotopy disjoint from [5] if there exists a map $g: S^m \to S^{2n}$ homotopic to f such that f and g are coincidence free; i.e., $f(x) \neq g(x)$ for all $x \in S^m$. The definition of homotopy disjoint by small deformation was introduced in [5]. An equivalent formulation in §7.2 of [6] affirms that (f, f) is homotopy disjoint by small deformation if, given $\varepsilon > 0$, there is an ε-homotopy $h_t: S^m \to S^{2n}$ from $f = h_0$ to $g = h_1$ such that f and g are coincidence free with $|h_t(x) - f(x)| < \varepsilon$ for all $x \in S^m$ and $0 \leqslant t \leqslant 1$. If (f, f) is homotopy disjoint by small deformation, then by

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definition (f, f) is homotopy disjoint. Moreover, the converse was proved for $m < 4n - 2$ by Lemma 1.14 of [5] or Theorem 2.2 of [9]. We use the notation of [11] for the homotopy of spheres. Any mapping $f : S^{30} \to S^{16}$ representing σ_{16}^2 has the property that (f, f) is homotopy disjoint, but cannot be homotopy disjoint by small deformation, according to Proposition 4.13 of [7]. Classically, σ_{16}^2 is a Kervaire invariant one element of order 2.

For any mapping $f : S^{4n-2} \to S^{2n}$ with $n > 1$, the Kervaire invariant of $[f]$, $KI([f])$, is the element of $Z/2$ obtained in the following manner. Via the adjoint construction, f produces a mapping

$$\hat{f} : S^{2n-2} \to \Omega^{2n} S^{2n} \to \Omega^{\infty} S^{\infty} = F. \tag{1}$$

Let $PL \to F \xrightarrow{c} F/PL$ denote the fibration associated to stable PL bundles and stable spherical fibrations. The classical Kervaire invariant morphism $KI : \pi_{2n-2}^s \to Z/2$, defined in terms of the Arf invariant of framed manifolds, can be identified with the morphism

$$c_* : \pi_{2n-2}(F) \to \pi_{2n-2}(F/PL) \tag{2}$$

for all even $n \geqslant 2$, according to [3] and Section 2 of [10]. Now c_* is the trivial morphism for all odd $n > 1$, since $\pi_{2n-2}(F/PL) \approx Z$ while $\pi_{2n-2}(F)$ is finite. By definition, $KI([f]) = [c \circ \hat{f}] = c_*([\hat{f}])$ for any mapping $f : S^{4n-2} \to S^{2n}$ with $n \geqslant 2$. If $KI([f]) = 1$, necessarily $n = 2^j$ for $j \geqslant 1$ by Theorem 7.1 of [3].

The Strong Kervaire Invariant One Problem conjectures the existence of mappings $f : S^{2^{j+1}-2} \to S^{2^j}$ for $j \geqslant 4$ such that $KI([f]) = 1$ and $2[f] = 0$. Theorem 2.1 affirms that mappings $f : S^{4n-2} \to S^{2n}$ for which (f, f) is homotopy disjoint, but cannot be homotopy disjoint by small deformation, are precisely the solutions to the Strong Kervaire Invariant One Problem. Theorem 9.1 of [4] presents five formulations equivalent to the Strong Kervaire Invariant One Problem. Perhaps the best known formulation equivalent to the existence of a class $\theta \in \pi_{2^j-2}^s$ of order 2 with $KI(\theta) = 1$ for $j > 1$ is the divisibility of the Whitehead square $[\iota_{2^j-1}, \iota_{2^j-1}]$ by 2, established in Corollary 3.2 of [1]. See [2] also.

The survey of coincidence theory in [6] provides more background material on self-coincidences of mappings. The material in this note was investigated further in [8].

2. Main result

This Note provides the following equivalent formulation:

Theorem 2.1. *Let $f : S^{4n-2} \to S^{2n}$ with $n > 4$ be any mapping such that (f, f) is homotopy disjoint. Then (f, f) cannot be homotopy disjoint by small deformation if and only if the class $[f]$ has Kervaire invariant one.*

The statements in the abstract follow from Theorem 2.1 in the following manner. Proposition 2.10 of [5] affirms that (f, f) is homotopy disjoint if and only if $A \circ f$ is homotopic to $f : S^{4n-2} \to S^{2n}$ with $n > 4$, where A denotes the antipodal map on S^{2n}. In other words, (f, f) is homotopy disjoint if and only if $2[f] = 0$ in $\pi_{4n-2}(S^{2n}) \equiv \pi_{2n-2}^s$. Given any $f : S^{4n-2} \to S^{2n}$ with $n > 4$ such that (f, f) is homotopy disjoint, but not homotopy disjoint by small deformation, then $KI([f]) = 1$ by Theorem 2.1. Necessarily, $2n = 2^j$ for some $j > 3$ by Theorem 7.1 of [3]. Thus $f : S^{2^{j+1}-2} \to S^{2^j}$ produces a solution $[f]$ to the Strong Kervaire Invariant One Problem. Conversely, any representative $g : S^{2^{j+1}-2} \to S^{2^j}$ with $j \geqslant 4$ of a solution $\theta \in \pi_{2^j-2}^s$ to the Strong Kervaire Invariant One Problem must have (g, g) homotopy disjoint since $2[g] = 0$. Moreover, (g, g) cannot be homotopy disjoint by small deformation by Theorem 2.1, since $KI([g]) = 1$.

Example 1. There exist maps $f : S^{2^{j+1}-2} \to S^{2^j}$ for $4 \leqslant j \leqslant 6$ such that (f, f) is homotopy disjoint, but not by small deformation, since the stable homotopy groups $\pi_{14}^s \cong Z/2 \oplus Z/2$, $\pi_{30}^s \cong Z/2 \oplus Z/3$ and $\pi_{62}^s \cong (Z/2)^2 \oplus Z/4 \oplus Z/3$ contain Kervaire invariant one classes of order 2.

Example 2. Theorem 2.1 clearly requires that $n > 4$. Any representative $f : S^6 \to S^4$ for η_4^2 and any representative $g : S^{14} \to S^8$ for ν_8^2 possess the property that both(f, f) and (g, g) are homotopy disjoint by small deformation, yet they represent Kervaire invariant one elements of order 2. This observation follows from the triviality of Whitehead products in H-spaces and the proof of Theorem 2.1.

3. Proof of Theorem 2.1

The classical *EHP* sequence follows from the homotopy exact sequence for the fibration $S^{2n-1} \to \Omega S^{2n} \to \Omega S^{4n-1}$. This produces a short exact sequence

$$0 \to Z/2 \to \pi_{4n-3}\big(S^{2n-1}\big) \xrightarrow{\Sigma} \pi_{4n-2}\big(S^{2n}\big) \to 0 \tag{3}$$

with the image of $Z/2$ being the subgroup generated by the Whitehead square $[\iota_{2n-1}, \iota_{2n-1}]$ for $n > 4$. Consequently, the sequence is split exact if and only if $[\iota_{2n-1}, \iota_{2n-1}]$ is not divisible by 2.

Propositions 2.13 and 2.16 of [5] affirm that (f, f) is homotopy disjoint by small deformation if and only if $f: S^{4n-2} \to S^{2n}$ admits a lifting to the Stiefel manifold $V_{2n+1,2}$ of orthonormal 2-frames in $\mathbb{R}^{2n+1}$. Equivalently, $\partial([f]) = 0$ where ∂ is the boundary operator in the homotopy exact sequence for the tangent sphere bundle fibration $S^{2n-1} \to V_{2n+1,2} \to S^{2n}$. Given $f: S^{4n-2} \to S^{2n}$ such that (f, f) is homotopy disjoint with $n > 4$, then $2[f] = 0$. Since $\partial(\iota_{2n}) = 2\iota_{2n-1}$, $\partial([f]) = [(2\iota_{2n-1}) \circ g] = 2[g] = 0$ whenever sequence [3] splits, with $[f] = \Sigma[g]$. Consequently, f lifts to $V_{2n+1,2}$ and so (f, f) is homotopy disjoint by small deformation.

Now the sequence (3) does not split precisely when $2n = 2^j$ for some $j > 3$ and there exists a class θ of order 2 and Kervaire invariant one in $\pi_{2^{j+1}-2}(S^{2^j})$. For any such class θ, every desuspension $\alpha \in \pi_{2^{j+1}-3}(S^{2^j-1})$ of θ has the property that $2\alpha = [\iota_{2^j-1}, \iota_{2^j-1}]$. For any representative $f: S^{2^{j+1}-2} \to S^{2^j}$ of any such class θ, (f, f) is homotopy disjoint since $2[f] = 0$. But (f, f) cannot be homotopy disjoint by small deformation, since $\partial(\theta) = 2\alpha \neq 0$.

Conversely, if $f: S^{2^{j+1}-2} \to S^{2^j}$ for $j > 3$ represents an element $[f]$ of order 2 such that (f, f) cannot be homotopy disjoint by small deformation, then necessarily $\partial([f]) = 2\alpha \neq 0$ where $[f] = \Sigma\alpha$. Consequently, α must have order 4 so $2\alpha = [\iota_{2^j-1}, \iota_{2^j-1}]$. Moreover, $[f]$ has Kervaire invariant one by [3] and by the well-known fact in [1] that the stable secondary operation $\Phi_{j-1,j-1}$, based on the relation $Sq^{2^{j-1}} Sq^{2^{j-1}} + \sum_{i=0}^{j-2} Sq^{2^j-2^i} Sq^{2^i} = 0$, detects a class $[f] \in \pi_{2^{j+1}-2}(S^{2^j})$ with $2[f] = 0$ if and only if $[\iota_{2^j-1}, \iota_{2^j-1}] = 2\alpha$ where $\Sigma\alpha = [f]$.

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