Differential Geometry

Floer homology for almost Hamiltonian isotopies

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Abstract

Seidel introduced a homomorphism from the fundamental group \( \pi_1(\text{Ham}(M)) \) of the group of Hamiltonian diffeomorphisms of certain compact symplectic manifolds \((M, \omega)\) to a quotient of the automorphism group \(\text{Aut}(HF_*(M, \omega))\) of the Floer homology \(HF_*(M, \omega)\). We prove a rigidity property: if two Hamiltonian loops represent the same element in \(\pi_1(\text{Diff}(M))\), then the image under the Seidel homomorphism of their classes in \(\pi_1(\text{Ham}(M))\) coincide. The proof consists in showing that Floer homology can be defined by using ‘almost Hamiltonian’ isotopies, i.e., isotopies that are homotopic relatively to endpoints to Hamiltonian isotopies.


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Résumé

Homologie de Floer pour les isotopies presques hamiltonniennes. Seidel a introduit un homomorphisme du groupe fondamental \( \pi_1(\text{Ham}(M)) \) du groupe des difféomorphismes Hamiltoniens de certaines variétés symplectiques compactes \((M, \omega)\) dans un quotient du groupe \(\text{Aut}(HF_*(M, \omega))\) des automorphismes de l’homologie de Floer \(HF_*(M, \omega)\). Nous démontrons que si deux lacets Hamiltoniens représentent le même élément dans \(\pi_1(\text{Diff}(M))\), alors les images par l’homomorphisme de Seidel de leurs classes dans \(\pi_1(\text{Ham}(M))\) coïncident (un phénomène de rigidité). La preuve consiste à montrer que l’homologie de Floer peut être définie en utilisant des isotopies presques Hamiltoniennes, c’est-à-dire des isotopies qui sont homotopes, relativement aux extrémités à des isotopies Hamiltoniennes.


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1. Introduction

A symplectic manifold is a manifold \(M\) which is equipped with a closed, non-degenerate 2-form \(\omega\). Non-degeneracy means that the interior product \(i_X\omega\) of a vector field \(X\) with \(\omega\) induces a bundle isomorphism from \(TM\) to \(T^*M\). Given \(H \in C^\infty(M \times \mathbb{R})\), this isomorphism defines a family of vector fields \(X_H\) on \(M\) by \(i_{X_H}\omega = dH_t\), where \(H_t(x) = H(x, t)\). Integration of this family of vector fields leads to an isotopy \(\theta^H = (\theta^H_t)\) of \(M\). We say that \(\theta^H\)

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is the Hamiltonian isotopy generated by $H$. The set of all time one maps of Hamiltonian isotopies forms a subgroup of $\text{Diff}(M)$, called the group of Hamiltonian diffeomorphisms, and it will be denoted $\text{Ham}(M)$ herein.

We are in general interested in the inclusion $i: \text{Ham}(M) \to \text{Diff}(M)$ and the induced map $i_\ast: \pi_1(\text{Ham}(M)) \to \pi_1(\text{Diff}(M))$. In [6], Seidel introduced an extension of $\pi_1(\text{Ham}(M))$, denoted $\widetilde{G}$, and a homomorphism $\sigma : \widetilde{G} \to \text{Aut}(H^\ast_{\text{Floer}}(M, \omega))$, where $H^\ast_{\text{Floer}}(M, \omega)$ is the Floer homology of $(M, \omega)$. (At this point, $\sigma$ is only well defined under certain conditions on $(M, \omega)$.) There is thus an induced homomorphism

$$\sigma : \pi_1(\text{Ham}(M)) \to \text{Aut}(H^\ast_{\text{Floer}}(M, \omega))/\theta.$$  

The main point of this Note is to sketch a proof of the following theorem (under the conditions necessary to define $\sigma$).

**Theorem 1.1.** Let $g_0$ and $g_1$ be smooth loops at id in $\text{Ham}(M)$, and let $[g_0]$ and $[g_1]$ be their respective classes in $\pi_1(\text{Ham}(M))$. Then, if $g_0$ and $g_1$ define the same element of $\pi_1(\text{Diff}(M))$, then $\sigma([g_0]) = \sigma([g_1])$.

In other words, this theorem says that $\ker i_\ast \subset \ker \sigma$. This gives more evidence for what has been called the ‘topological rigidity’ of the group $\text{Ham}(M)$ [4].

In order to prove this theorem, we observe that $\sigma$ is induced by a map $\hat{\sigma}$ with domain $G$ consisting of smooth loops at id in $\text{Ham}(M)$ (rather than $\pi_0(G) = \pi_1(\text{Ham}(M))$). We extend the domain of $\hat{\sigma}$ to the possibly larger group $D := \{\text{smooth loops in } \text{Diff}(M) \text{ at id which are homotopic to a Hamiltonian loop}\}$.

Elements of $D$ will be called almost Hamiltonian loops.

Theorem 1.1 is proved by showing that the extended map is well-defined on $\pi_0(D) = i_\ast(\pi_1(\text{Ham}(M)))$.

2. Floer homology and Seidel’s homomorphism [5,6]

Let $\mathcal{J}(M, S^1)$ denote the space of smooth time dependent 1-periodic almost complex structures on $M$. Let $\mathcal{J}(M, \omega, S^1) \subset \mathcal{J}(M, S^1)$ be the subspace of $\omega$-compatible almost complex structures. There are Chern classes $c_i \in H^{2i}(M, \mathbb{Z})$ associated to the symplectic manifold $(M^{2n}, \omega)$, and throughout this note, we will assume the semi-positive condition $W^+$ given in [6]. While the new ideas in this note do not depend on this assumption, it is thus far necessary in defining Seidel’s homomorphism.

Let $\tilde{LM}$ denote the space of smooth, contractible loops in $M$, and let $p: \tilde{LM} \to LM$ be the covering space given in [3]. For $H \in C^\infty(M \times S^1)$, let $\overline{\mathcal{P}}(H) \subset \tilde{LM}$ consist of 1-periodic contractible orbits of the Hamiltonian isotopy $\theta^H$ generated by the function $H$.

Let $(H, J) \in C^\infty(M \times S^1) \times \mathcal{J}(M, \omega, S^1)$ be a regular pair [2,3,6]. We denote by $H^\ast_{\text{Floer}}(M, \omega, H, J)$ the Floer homology groups defined using the pair $(H, J)$, which is essentially constructed using the set $\overline{\mathcal{P}}(H) := p^{-1}(\mathcal{P}(H))$.

Any $g \in G$ gives a map $g: \tilde{LM} \to \tilde{LM}$ by $(g \cdot x)(t) = g(x(t))$. Let $\widetilde{G}$ be the group of all lifts of elements of $G$ to homeomorphisms of $\tilde{LM}$. This gives an exact sequence of topological groups $1 \to \Gamma \to \widetilde{G} \to G \to 1$, where $\Gamma$ is the group of deck transformations of the covering map $p$.

For $g \in G$, a new regular pair $(H^8, J^8)$ is defined in such a way that $g \cdot \mathcal{P}(H^8) = \mathcal{P}(H)$.

After lifting to $\tilde{g} \in \tilde{G}$, this induces a map between the Floer chain complexes $CF^\ast_{\text{Floer}}(M, \omega, H^8)$ to $CF^\ast_{\text{Floer}}(M, \omega, H)$. The almost complex structure $J^8$ is defined so that this is a chain map, and this gives an isomorphism on the level of homology, which we still call $\tilde{g}$.

There is a continuation isomorphism $\Phi$ between the Floer homology groups defined using any two regular pairs. Let $\overline{\Phi}$ denote the continuation isomorphism between $H^\ast_{\text{Floer}}(M, \omega, H^8, J^8)$ and $H^\ast_{\text{Floer}}(M, \omega, H, J)$. Then we get an automorphism of $H^\ast_{\text{Floer}}(M, \omega, H, J)$ by

$$H^\ast_{\text{Floer}}(M, \omega, H, J) \xrightarrow{\Phi} H^\ast_{\text{Floer}}(M, \omega, H^8, J^8) \xrightarrow{\tilde{g}} H^\ast_{\text{Floer}}(M, \omega, H, J).$$  

This automorphism commutes with the continuation isomorphism to the Floer homology defined using any other regular pair, and if $g_0, g_1 \in G$ are connected through smooth Hamiltonian loops, then the corresponding isomorphisms agree. This describes a map

$$\hat{\sigma} : \pi_0(\widetilde{G}) \to \text{Aut}(H^\ast_{\text{Floer}}(M, \omega)).$$

After quotienting by the image of $\Gamma$ in $\text{Aut}(H^\ast_{\text{Floer}}(M, \omega))$, we obtain the map $\sigma$. 


3. Sketch of the proof of the main result

First notice that it is a near triviality to see that if two functions \( H^0 \) and \( H^1 \) generate the same Hamiltonian isotopy, then \( HF_*(M, \omega, H^0, J) = HF_*(M, \omega, H^1, J) \) (as long as both \((H^0, J)\) and \((H^1, J)\) are regular pairs). This means that Floer homology can be rephrased in terms of Hamiltonian isotopies.

The isotopy generated by the function \( H^\delta \) is given by \( \theta_t^{H^\delta} = g_t^{-1}\theta_t^H \). We define an action of \( D \) on the set of all isotopies of \( M \) by \((g \ast \theta)_t := g_t^{-1}\theta_t\), and in this language, \( \theta_t^{H^\delta} = g \ast \theta^H \).

Thus, we can rephrase \((2)\) as

\[
HF_*(M, \omega, \theta, J) \xrightarrow{\phi} HF_*(M, \omega, g \ast \theta, J^\delta) \xrightarrow{\tilde{\pi}} HF_*(M, \omega, \theta, J).
\]

This suggests a method of extending the domain of \( \hat{\sigma} \) to the group \( D \): we extend the space of available choices in defining \( HF_*(M, \omega) \) to include pairs \((g \ast \theta, J^\delta)\) for \( g \in D \) and Hamiltonian isotopies \( \theta \).

Notice that for \( g \in D \) and a Hamiltonian isotopy \( \theta, g \ast \theta \) is homotopic, relative endpoints, to a Hamiltonian isotopy. Such isotopies will be called almost Hamiltonian, and we denote the group of all almost Hamiltonian isotopies by \( \mathcal{I} \). There is an analogous extension \( 1 \rightarrow \mathcal{I} \rightarrow \mathcal{D} \rightarrow D \rightarrow 1 \), where \( \mathcal{D} \) consists of lifts of the action of \( D \) on \( \mathcal{L}M \) to homeomorphisms of \( \mathcal{L}M \).

Such isotopies may not preserve \( \omega \), so we need to adjust the compatibility requirements on \( J \).

For \( \psi \in \mathcal{I} \), let \( J_1^\psi (M, \omega, S^1) = \{ J \in J(M, S^1) \mid J_t \in J(M, (\psi_t^{-1})^\ast \omega) \} \). We then let \( \mathcal{F} \) denote all pairs \((\psi, J)\) with \( \psi \in \mathcal{I} \) and \( J \in J_1^\psi(M, \omega, S^1) \). It is best to think of \( \mathcal{F} \) as a bundle over \( \mathcal{I} \), with fiber over \( \psi \in \mathcal{I} \) given by \( J_1^\psi(M, \omega, S^1) \).

**Theorem 3.1.** There is a dense subset of \( \mathcal{F} \), denoted \( \mathcal{F}_{\text{reg}} \), such that for all pairs \((\psi, J) \in \mathcal{F}_{\text{reg}} \), we can define Floer homology groups \( HF_*(M, \omega, \psi, J) \). These groups are naturally independent of the choice of pair used to define them, and they recover the standard Floer homology groups \( HF_*(M, \omega) \).

The Floer homology groups \( HF_*(M, \omega, \psi, J) \) in this theorem are constructed as in the Hamiltonian case, but we replace the Hamiltonian function by the generated isotopy. More precisely, for \( \psi \in \mathcal{I} \), let \( X_1^\psi \) be the corresponding family of vector fields on \( M \) obtained by differentiation (see [1]), and set

\[
\mathcal{P}(\psi) := \{ x \in \mathcal{L}M \mid \dot{x}(t) = X_1^\psi (x(t)) \}, \quad \mathcal{P}(\psi) := p^{-1}(\mathcal{P}(\psi)).
\]

For certain (non-degenerate) isotopies, there is an index map \( \mu_\psi : \mathcal{P}(\psi) \to \mathbb{Z} \), which reduces to the Conley–Zehnder index in the Hamiltonian case. Notice that any homotopy from \( \psi \) to a Hamiltonian isotopy \( \theta \) provides an identification \( b : \mathcal{P}(\psi) \to \mathcal{P}(\theta) \). The index in the almost Hamiltonian case is well defined by setting \( \mu_\psi(c) = \mu_\theta(b(c)) \).

This graded set is used to create a chain complex, with a boundary operator defined as follows. For smooth \( u : \mathbb{R} \times S^1 \to M \), define \( \tilde{\partial}_\psi, J(u) \in C^\infty(u^* TM) \) by

\[
\tilde{\partial}_\psi, J(u) = \frac{\partial u}{\partial s} - J_t \left( \frac{\partial u}{\partial s} + X_1^\psi \right).
\]

For \( c_\pm \in \mathcal{P}(\psi) \), let \( \mathcal{M}(c_-, c_+, \psi, J) \) consist of all smooth maps \( u : \mathbb{R} \times S^1 \to M \) which satisfy \( \tilde{\partial}_\psi, J(u) = 0 \), and which lift to a map \( \tilde{u} : \mathbb{R} \to \mathcal{L}M \) with \( \lim_{s \to \pm \infty} \tilde{u}(s) = c_\pm \). There is an \( \mathbb{R} \)-action on \( \mathcal{M}(c_-, c_+, \psi, J) \) by translation in the \( s \)-variable, and we denote the quotient space by \( \mathcal{M}(c_-, c_+, \psi, J)/\mathbb{R} \).

The boundary operator in classical Floer homology consisted of counting the number of elements of \( \mathcal{M}(c_-, c_+, \psi, J)/\mathbb{R} \). We would like to do the same, so we show that if \( \mu_\psi(c_-) - \mu_\psi(c_+) = 1 \), then \( \mathcal{M}(c_-, c_+, \psi, J)/\mathbb{R} \) is finite.

In the Hamiltonian case (when \( \psi \) is a Hamiltonian isotopy), there are two basic reasons which allow us to count \( \mathcal{M}(c_-, c_+, \psi, J)/\mathbb{R} \). The first is the ellipticity of (4), and the second is a uniform bound on the energy \( E(u) = \int_{\mathbb{R} \times S^1} |\dot{u}_s(s, t)|^2 \). The ellipticity gives that \( \mathcal{M}(c_-, c_+, \psi, J) \) is a manifold, and the energy bound gives compactness (in the 0-dimensional case). The energy bound is proved by noticing that solutions to \( \tilde{\partial}_\psi, J(u) = 0 \) satisfy the equation \( \nabla a_H(\tilde{u}) = 0 \) for any lift \( \tilde{u} : \mathbb{R} \to \mathcal{L}M \), where \( a_H \) is the action functional. This implies that the energy of \( u \in \mathcal{M}(c_-, c_+, \psi, J) \) is given by \( E(u) = a_H(c_-) - a_H(c_+) \).
In the almost Hamiltonian case, we can establish the same facts: the term involving the vector field does not affect the ellipticity because it is of lower order. To give a bound on the energy, we decompose \((\psi, J) \in \mathcal{F}\) as \(g * \theta^H\) for some \(g \in D\) and Hamiltonian isotopy \(\theta^H\), and choose any lift \((g, \tilde{g}) \in \tilde{D}\). The pair \((\psi, J) \in \mathcal{F}\) induces a metric \(\tilde{h}\) on \(\tilde{LM}\). Then lifts \(\tilde{u}\) of solutions to \(\tilde{\partial}_{\psi, J}(u) = 0\) satisfy
\[
\tilde{h}\left(\nabla \tilde{g}^* a_H(\tilde{u})(s), \xi\right) = \tilde{g}^* (da_H)(\tilde{u}(s))(\xi) = \int_0^1 \psi_t^* \omega(\dot{x}(t) - X_{g^* \theta^H}(x(t)), J_t Dp \xi(t)) \, dt,
\]
where \(x(t) = u(s, t)\). But this is exactly \(\tilde{h}(\frac{du}{ds}(s), \xi)\). Thus, for \(u \in \mathcal{M}(c_-, c_+, \psi, J)\), \(E(u) = \tilde{g}^* a_H(c_-) - \tilde{g}^* a_H(c_+)\).

We then define a boundary operator via counting exactly as in the Hamiltonian case, and show that \(\partial^2 = 0\).

To prove the independence of the choice of pair, we define a homotopy of regular pairs as a map \(\Phi : \mathbb{R} \to \mathcal{F}\) which is fixed outside of \([-1, 1]\). A map between the corresponding chain complexes can be defined by counting cylinders which satisfy a two-parameter version of (4). These sets can be counted for the same reasons as described above.

Theorem 3.1 can now be used to extend the domain of \(\tilde{\sigma}\) to \(\tilde{D}\) by (3). The final step is proving that this map is well defined on \(\pi_0(\tilde{D})\). This is accomplished by combining Seidel’s proof with the above ideas.

References