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Mathematical Problems in Mechanics

Beltrami's solutions of general equilibrium equations in continuum mechanics

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Abstract

M. Gurtin has proved that the Beltrami representation, $\mathbf{S} = \operatorname{rot}\operatorname{rot}\mathbf{A}$, of a smooth, divergence-free stress tensor in a smooth domain, is verified if and only if $\mathbf{S}$ is self-equilibrated. Here, Gurtin's conditions are extended to the case of a bounded domain with a Lipschitz-continuous boundary, for a tensor field $\mathbf{S} \in L^2(\Omega; M^3_{\text{sym}})$. We apply this result to obtain an extension of the Saint Venant's equations of compatibility to non necessarily simply-connected domains. ***To cite this article: G. Geymonat, F. Krasucki, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Solutions de Beltrami des équations d'équilibre de la mécanique des milieux continus. M. Gurtin a montré que la représentation de Beltrami, $\mathbf{S} = \operatorname{rot}\operatorname{rot}\mathbf{A}$, d'un champ régulier de contraintes à divergence nulle dans un ouvert à bord régulier est vérifiée si et seulement si $\mathbf{S}$ est auto-équilibré. Les conditions données par Gurtin sont étendues au cas d'un ouvert à bord Lipschitzien pour un champ $\mathbf{S} \in L^2(\Omega; M^3_{\text{sym}})$. Par application de ce résultat on trouve une extension des conditions de compatibilité de Saint Venant aux domaines non nécessairement simplement connexes. ***Pour citer cet article : G. Geymonat, F. Krasucki, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Soit Ω un domaine ouvert connexe et borné de $\mathbb{R}^3$ à frontière Lipschitzienne $\partial\Omega$. On note $\{x_1, x_2, x_3\}$ les coordonnées d'un point par rapport au repère orthonormé $\{\mathbf{e}_i\}$, $(i = 1, 2, 3)$. Les composantes d'un champ de vecteurs $\mathbf{v}$ sont notés v_i et celles d'un tenseur du second ordre $\mathbf{S}$ par S_{ij}. On utilise la convention de l'indice répété. On note $\operatorname{rot}\mathbf{E}$ le tenseur de composantes : $(\operatorname{rot}\mathbf{E})_{ij} = \epsilon_{ipk} E_{jk,p}$. La virgule indique la dérivation par rapport à x et ϵ_{ipk} est le tenseur des permutations. M^3_{sym} est l'espace vectoriel des tenseurs du second ordre symétriques. On définit

$$\Sigma = \left\{\mathbf{S} \in L^2\left(\Omega; M^3_{\text{sym}}\right);\ \operatorname{div}\mathbf{S} \in \left(L^2(\Omega)\right)^3\right\}.$$

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En suivant l'approche de Lions–Magenes [9], l'application linéaire $\mathbf{S} \mapsto \Gamma_{\mathbf{n}}(\mathbf{S}) = (\mathbf{S} \cdot \mathbf{n})|_{\partial\Omega}$ est définie et continue de Σ dans $(H^{-1/2}(\partial\Omega))^3$ et on a la formule de Green (1), où $\langle\cdot,\cdot\rangle_{\partial\Omega}$ est le produit de dualité entre $(H^{-1/2}(\partial\Omega))^3$ et $(H^{1/2}(\partial\Omega))^3$ et $\mathbf{E}(\mathbf{v}) = \frac{1}{2}(\nabla\mathbf{v}^{\mathrm{T}} + \nabla\mathbf{v})$. Dans [5] on montre que $\mathfrak{D}(\Omega; M^3_{\mathrm{sym}})$ est dense dans

$$\mathrm{Ker}(\Gamma_{\mathbf{n}}) = \left\{\mathbf{S} \in L^p\left(\Omega; M^3_{\mathrm{sym}}\right);\ \mathrm{div}\,\mathbf{S} \in \left(L^p(\Omega)\right)^3;\ \Gamma_{\mathbf{n}}(\mathbf{S}) = 0 \text{ sur } \partial\Omega\right\}$$

et que $\mathcal{V} = \{\mathbf{S} \in \mathfrak{D}(\Omega; M^3_{\mathrm{sym}});\ \mathrm{div}\,\mathbf{S} = 0 \text{ in } \Omega\}$ est dense dans Σ_{ad} défini par :

$$\Sigma_{\mathrm{ad}} = \left\{\mathbf{S} \in L^2\left(\Omega; M^3_{\mathrm{sym}}\right);\ \mathrm{div}\,\mathbf{S} = 0 \text{ dans } \Omega \text{ et } \Gamma_{\mathbf{n}}(\mathbf{S}) = 0 \text{ sur } \partial\Omega\right\}.$$

On dit qu'un champ de tenseurs du second ordre, $\mathbf{A}$, régulier et symétrique est une fonction de contrainte s'il existe un operateur différentiel $\mathcal{L}$ tel que $\mathbf{S} = \mathcal{L}(\mathbf{A}) = \mathbf{S}^{\mathrm{T}}$ vérifie $\mathrm{div}\,\mathbf{S} = 0$. Dans le cas bidimensionel, la solution des équations d'équilibre est exprimée à l'aide d'une fonction scalaire, la fonction d'Airy. Beltrami a observé, voir e.g. [8] pour une présentation générale, que le cas bidimensionnel et toutes les extensions au cas tridimensionel sont des choix particuliers des solutions définies par la Proposition 2.1.

Par ailleurs, il existe des champs de tenseurs à divergence nulle, qui ne sont pas représentés par une solution de Beltrami, c'est à dire qui ne dérivent pas d'une fonction de contrainte. Dans [8], une condition nécessaire et suffisante est donnée pour des domaines réguliers. Cette condition est étendue au cas plus général suivant :

Théorème 0.1 (Beltrami's completeness). *Soient γ_q les composantes connexes de $\partial\Omega$, $q = 0, \ldots, Q$.*

(i) *Soit $\mathbf{S} \in L^2(\Omega; M^3_{\mathrm{sym}})$. Alors $\mathbf{S} = \mathrm{rot}\,\mathrm{rot}\,\mathbf{A}$, avec $\mathbf{A} \in H^2(\Omega; M^3_{\mathrm{sym}})$ si et seulement si $\mathrm{div}\,\mathbf{S} = 0$, et respecte les conditions* (3) *et* (4) *où les composantes du vecteur $\mathbf{p}^i$ sont : $p^i_j = -\epsilon_{ijk}x_k$;*

(ii) *$\mathbf{S} \in \Sigma_{\mathrm{ad}}$, si et seulement si $\mathbf{S} = \mathrm{rot}\,\mathrm{rot}\,\mathbf{A}$, avec $\mathbf{A} \in H^2_0(\Omega; M^3_{\mathrm{sym}})$.*

(i) La démonstration s'appuie sur le Théorème 2.3 appliqué de façon réitérée aux vecteurs formés par les lignes des tenseurs concernés. On utilise de façon essentielle l'identité (8).

(ii) La démonstration est une conséquence de la densité de $\mathfrak{D}(\Omega; M^3_{\mathrm{sym}})$ dans $H^2_0(\Omega; M^3_{\mathrm{sym}})$ et de celle de $\mathcal{V}$ dans Σ_{ad}.

Cette caractérisation des solutions de Beltrami permet d'étendre un théorème de [2], au cas non simplement connexe.

Théorème 0.2. *Soit $\mathbf{E}$ un champ de tenseurs symétriques dans $L^2(\Omega; M^3_{\mathrm{sym}})$ vérifiant les relations de compatibilité de Saint Venant,* $\mathrm{rot}\,\mathrm{rot}\,\mathbf{E} = 0$ *dans $H^{-2}(\Omega; M^3_{\mathrm{sym}})$. Alors il existe un vecteur $\mathbf{v} \in (H^1(\Omega))^3$ vérifiant $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{v}^{\mathrm{T}} + \nabla\mathbf{v})$.*

1. Introduction

Let Ω be an open, connected and bounded domain in $\mathbb{R}^3$ with a Lipschitz-continuous boundary $\partial\Omega$. Relative to an orthonormal Cartesian basis $\{\mathbf{e}_i\}$, $(i = 1, 2, 3)$, the coordinates of a generic point will be denoted by $\{x_1, x_2, x_3\}$, the components of a vector field $\mathbf{v}$ by v_i and the components of a second-order tensor field $\mathbf{S}$, by S_{ij}. The summation convention with respect to the repeated indices is used.

Let $\mathbf{E}$ be a smooth symmetric second-order tensor field. We denote by $\mathrm{rot}\,\mathbf{E}$ the tensor whose components are: $(\mathrm{rot}\,\mathbf{E})_{ij} = \epsilon_{ipk}E_{jk,p}$. The commas stand for partial derivatives with respect to x and ϵ_{ipk} denotes the alternator. We denote by M^3_{sym} the vector space of symmetric second-order tensors. Let us define:

$$\Sigma = \left\{\mathbf{S} \in L^2\left(\Omega; M^3_{\mathrm{sym}}\right);\ \mathrm{div}\,\mathbf{S} \in \left(L^2(\Omega)\right)^3\right\}.$$

Following the well-known approach of Lions and Magenes [9], the linear map $\mathbf{S} \mapsto \Gamma_{\mathbf{n}}(\mathbf{S}) = (\mathbf{S} \cdot \mathbf{n})|_{\partial\Omega}$ is well-defined and continuous from Σ to $(H^{-1/2}(\partial\Omega))^3$. Moreover, [6], for every $\mathbf{S} \in \Sigma$ and every $\mathbf{v} \in (H^1(\Omega))^3$, the following Green's formula holds:

$$\int_\Omega \mathbf{E}(\mathbf{v}) : \mathbf{S}\,\mathrm{d}\Omega + \int_\Omega \mathrm{div}\,\mathbf{S} \cdot \mathbf{v}\,\mathrm{d}\Omega = \left\langle\Gamma_{\mathbf{n}}(\mathbf{S}), \mathbf{v}\right\rangle_{\partial\Omega}, \tag{1}$$

where $\langle\cdot,\cdot\rangle_{\partial\Omega}$ denotes the duality pairing between $(H^{-1/2}(\partial\Omega))^3$ and $(H^{1/2}(\partial\Omega))^3$ and $\mathbf{E}(\mathbf{v})=\frac{1}{2}(\nabla\mathbf{v}^{\mathrm{T}}+\nabla\mathbf{v})$. In [5] we have proved that $\mathfrak{D}(\Omega;M^3_{\mathrm{sym}})$ is dense in

$$\mathrm{Ker}(\Gamma_{\mathbf{n}})=\left\{\mathbf{S}\in L^p\left(\Omega;M^3_{\mathrm{sym}}\right);\ \mathrm{div}\,\mathbf{S}\in\left(L^p(\Omega)\right)^3;\ \Gamma_{\mathbf{n}}(\mathbf{S})=0 \text{ on } \partial\Omega\right\}$$

and that $\mathcal{V}=\{\mathbf{S}\in\mathfrak{D}(\Omega;M^3_{\mathrm{sym}});\ \mathrm{div}\,\mathbf{S}=0 \text{ in } \Omega\}$ is dense in Σ_{ad} defined by:

$$\Sigma_{\mathrm{ad}}=\left\{\mathbf{S}\in L^2\left(\Omega;M^3_{\mathrm{sym}}\right);\ \mathrm{div}\,\mathbf{S}=0 \text{ in } \Omega \text{ and } \Gamma_{\mathbf{n}}(\mathbf{S})=0 \text{ on } \partial\Omega\right\}.$$

Green's formula (1) implies that $\int_\Omega \mathbf{E}(\mathbf{v}):\mathbf{S}\,\mathrm{d}\Omega=0$ for every $\mathbf{S}\in\Sigma_{\mathrm{ad}}$ and for every $\mathbf{v}\in(H^1(\Omega))^3$. The following extension of Ting's and Donati's theorems proved in [5] gives also the reciprocal statement:

Theorem 1.1. *Let be* $\mathbf{E}$ *in* $L^2(\Omega;M^3_{\mathrm{sym}})$. *Then,*

$$\int_\Omega \mathbf{E}:\mathbf{S}\,\mathrm{d}\Omega=0,\quad \textit{for every } \mathbf{S}\in\Sigma_{\mathrm{ad}} \tag{2}$$

if and only if there exists a vector $\mathbf{v}\in(H^1(\Omega))^3$ *satisfying* $\mathbf{E}(\mathbf{v})=\frac{1}{2}(\nabla\mathbf{v}^{\mathrm{T}}+\nabla\mathbf{v})$.

A similar result has been simultaneously obtained by Ph.G. Ciarlet, L. Gratie and S. Kesavan, albeit by different proofs, [3].

2. Completeness of Beltrami's solution

A regular symmetric tensor field $\mathbf{A}$ is said a stress function when there exists a differential operator $\mathcal{L}$ such that $\mathbf{S}=\mathcal{L}(\mathbf{A})=\mathbf{S}^{\mathrm{T}}$ verifies $\mathrm{div}\,\mathbf{S}=0$. In the two-dimensional case, one can express the solution of equations of equilibrium in terms of the so-called Airy's scalar function. Beltrami observed, see e.g. [8] for a general presentation of this topic, that this two-dimensional case and all the generalisations to the three-dimensional case are special choices of now called Beltrami's solution defined by the following proposition:

Proposition 2.1. *Let* $\mathbf{A}$ *a tensor field of class* $C^3(\Omega;M^3_{\mathrm{sym}})$ *and let* $\mathbf{S}=\mathrm{rot}\,\mathrm{rot}\,\mathbf{A}$; *then* $\mathrm{div}\,\mathbf{S}=0$ *and* $\mathbf{S}=\mathbf{S}^{\mathrm{T}}$.

Let us remark that there exist stress fields $\mathbf{S}$ that do not admit a representation as a Beltrami's solution. Therefore it may be of interest to find sufficient conditions on $\mathbf{S}$ in order that such a representation be true; such conditions are called completeness conditions. Gurtin [8] proves that for a smooth domain Ω a sufficient condition is the nullity of the resultant force and of the moment on each closed regular surface contained in Ω. Such stress fields are called self-equilibrated. A global condition of nullity was given for Airy's stress function in general Lipschitz domain in [4]. This condition is extended to the general Beltrami's solution in:

Theorem 2.2 (Beltrami's completeness). *Let be* γ_q *the connected components of* $\partial\Omega$, $q=0,\dots,Q$.

(i) *Let be* $\mathbf{S}\in L^2(\Omega;M^3_{\mathrm{sym}})$. *Then* $\mathbf{S}=\mathrm{rot}\,\mathrm{rot}\,\mathbf{A}$, *where* $\mathbf{A}\in H^2(\Omega;M^3_{\mathrm{sym}})$ *if and only if* $\mathrm{div}\,\mathbf{S}=0$, *and for* $i=1,2,3$, $q=0,\dots,Q$

$$\left\langle\Gamma_{\mathbf{n}}(\mathbf{S}),\mathbf{e}_i\right\rangle_{\gamma_q}=0, \tag{3}$$

$$\left\langle\Gamma_{\mathbf{n}}(\mathbf{S}),\mathbf{p}^i\right\rangle_{\gamma_q}=0, \tag{4}$$

where the components of the vector $\mathbf{p}^i$ *are*: $p^i_j=-\epsilon_{ijk}x_k$;

(ii) $\mathbf{S}\in\Sigma_{\mathrm{ad}}$, *if and only if* $\mathbf{S}=\mathrm{rot}\,\mathrm{rot}\,\mathbf{A}$, *where* $\mathbf{A}\in H^2_0(\Omega;M^3_{\mathrm{sym}})$.

Let us remark that (3) and (4) are global conditions of nullity of the resultant force and of the moment.

Theorem 2.2(i) gives the characterization of $\mathrm{Im}(\mathcal{B})$ where the map $\mathcal{B}$ from $H^2(\Omega;M^3_{\mathrm{sym}})$ into $L^2(\Omega;M^3_{\mathrm{sym}})$ is defined by: $\mathcal{B}:\mathbf{A}\to\mathcal{B}(\mathbf{A})=\mathrm{rot}\,\mathrm{rot}\,\mathbf{A}$.

The proof is based on the following theorem (see e.g. [7,1]):

Theorem 2.3.

(i) *For any* $\mathbf{u} \in (L^2(\Omega))^3$ *such that* $\operatorname{div}\mathbf{u} = 0$ *in* Ω *and*

$$\langle \mathbf{u}\cdot\mathbf{n}, 1\rangle_{\gamma_q} = 0 \quad \text{for } q = 0,\ldots,Q \tag{5}$$

there exists $\Psi \in (H^1(\Omega))^3$ *such that*:

$$\mathbf{u} = \operatorname{rot}\Psi \quad \text{in } \Omega \quad \text{and} \quad \operatorname{div}\Psi = 0 \quad \text{in } \Omega. \tag{6}$$

(ii) *Conversely, for any* $\Psi \in (H^1(\Omega))^3$*, the function* $\mathbf{u} = \operatorname{rot}\Psi$ *satisfies* $\operatorname{div}\mathbf{u} = 0$ *in* Ω *and* (5).
(iii) *Moreover, if* $\mathbf{u} \in (H^1(\Omega))^3$*, then* $\Psi \in (H^2(\Omega))^3$.

Proof of Theorem 2.2. (i) Let $\mathbf{A} \in H^2(\Omega; M^3_{\mathrm{sym}})$, then $\mathbf{S} = \operatorname{rot}\operatorname{rot}\mathbf{A} = \mathbf{S}^{\mathrm{T}}$ is a symmetric tensor field and satisfies $\operatorname{div}\mathbf{S} = 0$. Let us define $\mathbf{W} = \operatorname{rot}\mathbf{A} \in H^1(\Omega; M^3)$. The proof of (3), (4) follows from Theorem 2.3. More precisely, given a tensor $\Phi \in H^1(\Omega; M^3)$, let $\mathbf{R} = \operatorname{rot}\Phi$, then the vector $\mathbf{r}^i$ formed by the ith line of the tensor $\mathbf{R}^{\mathrm{T}}$ verifies $\mathbf{r}^i = \operatorname{rot}\Phi^i$, where Φ^i is formed by the ith line of the tensor Φ. The statement (ii) of Theorem 2.3 implies then:

$$\big\langle \Gamma_{\mathbf{n}}\big((\operatorname{rot}\Phi)^{\mathrm{T}}\big), \mathbf{e}_i\big\rangle_{\gamma_q} = \big\langle \Gamma_{\mathbf{n}}\big(\mathbf{R}^{\mathrm{T}}\big), \mathbf{e}_i\big\rangle_{\gamma_q} = \big\langle \mathbf{r}^i\cdot\mathbf{n}, 1\big\rangle_{\gamma_q} = 0 \quad \text{for } q = 0,\ldots,Q. \tag{7}$$

If one takes $\Phi = \mathbf{W}$, one obtains the global boundary conditions (3). In order to verify the global boundary conditions (4) we use the following identity:

$$\big\langle \Gamma_{\mathbf{n}}\big((\operatorname{rot}\mathbf{W})^{\mathrm{T}}\big), \mathbf{p}^i\big\rangle_{\gamma_q} = \big\langle \Gamma_{\mathbf{n}}\big(\operatorname{rot}(\mathbf{PW})\big)^{\mathrm{T}}, \mathbf{e}_i\big\rangle_{\gamma_q} + \big\langle \Gamma_{\mathbf{n}}\big(\mathbf{W}^{\mathrm{T}}\big), \mathbf{e}_i\big\rangle_{\gamma_q} - \big\langle \Gamma_{\mathbf{n}}\big(\operatorname{tr}(\mathbf{W})\mathbf{I}\big), \mathbf{e}_i\big\rangle_{\gamma_q}. \tag{8}$$

Since $\operatorname{tr}(\mathbf{W}) = 0$, (4) is obtained applying (7) with $\Phi = \mathbf{PW}$ and with $\Phi = \mathbf{A}$.

Conversely, let $\mathbf{S} \in L^2(\Omega; M^3_{\mathrm{sym}})$ satisfying the global conditions (3) and (4) and such that $\operatorname{div}\mathbf{S} = 0$. We can apply the statement (i) of the Theorem 2.3 to the vectors $\mathbf{s}^i$ formed by the ith line of the tensor $\mathbf{S}$. So, there exist some vectors $\Psi^i \in (H^1(\Omega))^3$ such that: $\mathbf{s}^i = \operatorname{rot}\Psi^i$ and $\operatorname{div}\Psi^i = 0$ in Ω. We define the tensor $\mathbf{W}$ whose lines are the vectors Ψ^i and the tensor $\mathbf{B} = \mathbf{W}^{\mathrm{T}} - \operatorname{tr}(\mathbf{W})\mathbf{I}$. It follows that $\mathbf{S}^{\mathrm{T}} = \operatorname{rot}\mathbf{W}$, $\mathbf{B} \in H^1(\Omega; M^3)$ and $\operatorname{div}\mathbf{B} = 0$. In order to apply the statement (i) of the Theorem 2.3 to the vectors $\mathbf{b}^i$ formed by the ith line of the tensor $\mathbf{B}$, we have to verify that the condition (5) is satisfied. For this, we use the formula (8) and the global condition (4). So, there exist some vectors $\mathbf{c}^i \in (H^2(\Omega))^3$ such that $\mathbf{b}^i = \operatorname{rot}\mathbf{c}^i$. We can define the tensor $\mathbf{C} \in H^2(\Omega; M^3)$ whose lines are the vectors $\mathbf{c}^i$. We remark that $\mathbf{W} = \mathbf{B}^{\mathrm{T}} + \operatorname{tr}(\mathbf{W})\mathbf{I} = \operatorname{rot}\mathbf{C} + \operatorname{tr}(\mathbf{W})\mathbf{I}$ and hence $\mathbf{S}^{\mathrm{T}} = \operatorname{rot}\mathbf{W} = \operatorname{rot}\operatorname{rot}\mathbf{C} + \operatorname{rot}(\operatorname{tr}(\mathbf{W})\mathbf{I})$ and $\mathbf{S} = \frac{\mathbf{S}^{\mathrm{T}}+\mathbf{S}}{2} = \operatorname{rot}\operatorname{rot}\frac{\mathbf{C}^{\mathrm{T}}+\mathbf{C}}{2} = \operatorname{rot}\operatorname{rot}\mathbf{A}$, with $\mathbf{A} \in H^2(\Omega; M^3_{\mathrm{sym}})$.

(ii) Since (i) implies that $\operatorname{Im}(\mathcal{B})$ is closed in $L^2(\Omega; M^3_{\mathrm{sym}})$, the statement follows from the density of $\mathfrak{D}(\Omega; M^3_{\mathrm{sym}})$ in $H^2_0(\Omega; M^3_{\mathrm{sym}})$ and of $\mathcal{V}$ in Σ_{ad}. □

The completeness of the Beltrami's solution allows to give the following extension of a result of Ph.G. Ciarlet and P. Ciarlet Jr., [2], to non necessarily simply-connected domains.

Theorem 2.4. *Let* $\mathbf{E}$ *a second order symmetric tensor field in* $L^2(\Omega; M^3_{\mathrm{sym}})$ *satisfying the compatibility relations* $\operatorname{rot}\operatorname{rot}\mathbf{E} = 0$ *in* $H^{-2}(\Omega; M^3_{\mathrm{sym}})$*. Then there exists a vector* $\mathbf{v} \in (H^1(\Omega))^3$ *satisfying the strain–displacement relations* $\mathbf{E} = \frac{1}{2}(\nabla\mathbf{v}^{\mathrm{T}} + \nabla\mathbf{v})$.

Proof. Using the derivation in distribution sense, the compatibility relations in $H^{-2}(\Omega; M^3_{\mathrm{sym}})$ mean, that:

$$0 = \langle \operatorname{rot}\operatorname{rot}\mathbf{E}, \mathbf{A}\rangle = \int_\Omega \mathbf{E} : \operatorname{rot}\operatorname{rot}\mathbf{A}\,\mathrm{d}\Omega \tag{9}$$

for every $\mathbf{A} \in H^2_0(\Omega; M^3_{\mathrm{sym}})$ where $\langle\cdot\,,\,\cdot\rangle$ denotes the duality pairing between $H^{-2}(\Omega; M^3_{\mathrm{sym}})$ and $H^2_0(\Omega; M^3_{\mathrm{sym}})$. Let then $\mathbf{S} = \operatorname{rot}\operatorname{rot}\mathbf{A}$, so that $\mathbf{S} \in \{\mathbf{S} \in L^2(\Omega; M^3_{\mathrm{sym}});\ \operatorname{div}\mathbf{S} = 0\}$. From the density of $\mathfrak{D}(\Omega; M^3_{\mathrm{sym}})$ in $H^2_0(\Omega; M^3_{\mathrm{sym}})$ it follows that $\mathbf{S} \in \Sigma_{\mathrm{ad}}$. One can then deduce from Theorem 1.1 that there exists a vector $\mathbf{v} \in (H^1(\Omega))^3$ satisfying the strain–displacement relations if one can prove:

$$\operatorname{rot}\operatorname{rot}\big(H_0^2\big(\Omega; M^3_{\mathrm{sym}}\big)\big) = \Sigma_{\mathrm{ad}}. \tag{10}$$

This is exactly the second statement of Theorem 2.2. □

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