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Numerical Analysis

A superconvergent projection method for nonlinear compact operator equations

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Abstract

We propose a method based on projections for approximating fixed points of a compact nonlinear operator. Under the same assumptions as in the Galerkin method, the proposed solution is shown to converge faster than the Galerkin solution. ***To cite this article: L. Grammont, R. Kulkarni, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Une méthode de projection superconvergente pour les équations d'opérateurs nonlinéaires compacts. Nous proposons une méthode, basée sur une projection, pour approcher les points fixes localement uniques d'un opérateur compact. Cette méthode présente un avantage par rapport aux méthodes de Galerkin et de Galerkin itérée étudiées par K.E. Atkinson and F.A. Potra : on n'a pas besoin de conditions supplémentaires pour obtenir la superconvergence de la solution approchée vers la solution exacte. ***Pour citer cet article : L. Grammont, R. Kulkarni, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Consider the following nonlinear operator equation

$$x = \mathcal{K}(x), \tag{1}$$

where $\mathcal{K}$ is a compact operator defined on $\overline{D}$, D being an open set of a real or complex Banach space $\mathcal{X}$.

We are interested in the evaluation of fixed points x^* of $\mathcal{K}$. In this note we investigate the use of a new projection method to approximate these fixed points.

We assume that the operator $\mathcal{K}$ has the following properties.

(H1) (i) Eq. (1) has a fixed point in D,
(ii) $\mathcal{K}$ is Fréchet differentiable on D and $D\mathcal{K}$ is q-Lipschitz in D,
(iii) 1 is not in the spectrum of $L = D\mathcal{K}(x^*)$.

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The condition (iii) implies that x^* is isolated as a fixed point of $\mathcal{K}$.

Let $\mathcal{X}_n$ be a sequence of finite dimensional subspaces of $\mathcal{X}$ and $P_n:\mathcal{X}\to\mathcal{X}_n$ a sequence of projections converging pointwise to the identity operator. In the classical Galerkin method, (1) is approximated by the following equation in $\mathcal{X}_n$:

$$x_n^G = P_n\mathcal{K}\big(x_n^G\big). \tag{2}$$

This method was analyzed by Krasnosel'skii and Zabreiko [4] and by Atkinson and Potra [2]. It has been proved that, if x^* is an isolated fixed point, then for sufficiently large n, equation (2) has a unique solution $x_n^G\in\mathcal{X}_n\cap D$ which converges to x^*. Also, the rate of convergence is the same as that of $P_n x^*$ to x^*.

In [2], Atkinson and Potra investigate the iterated Galerkin method given by

$$x_n^S = \mathcal{K}\big(x_n^G\big). \tag{3}$$

If $\mathcal{X}$ is an Hilbert space and P_n is the orthogonal projection, then in [2], x_n^S is shown to be superconvergent to x^* in the following sense.

$$\frac{\|x_n^S - x^*\|}{\|P_n x^* - x^*\|}\mapsto 0 \quad \text{as } n\to\infty.$$

In general, the iterated Galerkin approximation is superconvergent if and only if

$$\frac{\|D\mathcal{K}(x^*)(I-P_n)x^*\|}{\|(I-P_n)x^*\|}\to 0 \quad \text{as } n\to\infty. \tag{4}$$

This condition may fail as is seen in the following example:

Let $\mathcal{X}=L^\infty[0,1]$ and $\mathcal{X}_n$ the space of piecewise constant functions with respect to the uniform partition

$$0<\frac{1}{n}<\frac{2}{n}<\cdots<\frac{n}{n}=1. \tag{5}$$

Let $P_n: C[0,1]\to\mathcal{X}_n$ be the interpolatory map at the points $t_i^{(n)}=\frac{3i+1}{3n}$, $i=0,\ldots,n-1$. The domain of definition of P_n can be extended to $L^\infty[0,1]$ as in [1]. Let $\mathcal{K}$ be the following nonlinear operator defined on $L^\infty[0,1]$ with the range in $C[0,1]$:

$$\mathcal{K}(x)(s)=\left(\int_0^1 x(t)\,\mathrm{d}t+1\right)s, \quad s\in[0,1]. \tag{6}$$

Note that $\mathcal{K}$ is compact, Fréchet differentiable and for all $x\in L^\infty[0,1]$,

$$\big(D\mathcal{K}(x)\cdot h\big)(s)=\left(\int_0^1 h(t)\,\mathrm{d}t\right)s, \quad \forall s\in[0,1].$$

It is easy to show that (1) has a unique solution in $C[0,1]$: $x^*(s)=2s$ for all $s\in[0,1]$. Also 1 is not an eigenvalue of $D\mathcal{K}(x^*)$.

We have

$$\big\|D\mathcal{K}(x^*)(I-P_n)(x^*)\big\|_\infty=\frac{1}{3n}, \qquad \big\|(I-P_n)(x^*)\big\|_\infty\leqslant\frac{4}{3n}$$

and hence

$$\frac{\|D\mathcal{K}(x^*)(I-P_n)(x^*)\|_\infty}{\|(I-P_n)(x^*)\|_\infty}\geqslant\frac{1}{4}.$$

In this Note, we adapt to the nonlinear case the method introduced by Kulkarni in [5] and [6] for a linear operator equation and the associated eigenvalue problem. In contrast to the iterated Galerkin solution, we show that the proposed solution is superconvergent to the exact solution without any additional conditions such as (4).

2. A projection method

We define

$$\mathcal{K}_n^K(x) = P_n\mathcal{K}(x) + \mathcal{K}(P_n x) - P_n\mathcal{K}(P_n x) \tag{7}$$

and approximate (1) by

$$x = \mathcal{K}_n^K(x). \tag{8}$$

Remark 1. In [5] and [6], Kulkarni proposed to approximate a linear compact operator T by

$$T_n^K = P_n T P_n + (I - P_n)T P_n + P_n T(I - P_n).$$

Since T is linear,

$$T_n^K = P_n T + T P_n - P_n T P_n.$$

The last equality may not hold when T is nonlinear.

We first state preliminary lemmas which are needed in proving the main theorem. By assumption, x^* is a locally unique fixed point of $\mathcal{K}$. Let $\mathcal{V}$ be a neighbourhood of x^* in which (1) has a unique solution. Since P_n converges to the Identity operator pointwise, there is a constant p such that $\|P_n\| \leqslant p$.

Lemma 2.1. *Let $\delta > 0$ and $\varepsilon > 0$ be such that $B(x^*, \delta)$ and $B(x^*, \delta p + \varepsilon)$ are in $\mathcal{V}$. Then there exists a positive integer n_0 such that for $n \geqslant n_0$ and $x \in B(x^*, \delta)$, $P_n x \in \mathcal{V}$.*

Lemma 2.2. *For n large enough, $\mathcal{K}_n^K$ is Fréchet differentiable on $B(x^*, \delta)$, 1 is not in the spectrum of $D\mathcal{K}_n^K(x^*)$ and*

$$\left\|\left(I - D\mathcal{K}_n^K(x^*)\right)^{-1}\right\| \leqslant 2c,$$

where $c = \|(I - L)^{-1}\|$.

Lemma 2.3. *For n large enough, $D\mathcal{K}_n^K$ is Lipschitz continuous on $B(x^*, \delta)$, with Lipschitz constant $m \leqslant qp + qp^2(1 + p)$.*

We now prove our main result about the local existence and uniqueness of the fixed point of $\mathcal{K}_n^K$ defined by (7) and give an estimation of its rate of convergence. The proof is based on the contraction mapping principle. Let

$$a_n = \left\|(I - P_n)L\right\| \quad \text{and} \quad r_n = \frac{\|\mathcal{K}(P_n x^*) - \mathcal{K}(x^*) - L(P_n x^* - x^*)\|}{\|P_n x^* - x^*\|}.$$

As $L = D\mathcal{K}(x^*)$ is compact, a_n tends to 0 and as $\mathcal{K}$ is Fréchet differentiable, r_n tends to 0.

Theorem 2.4. *Under the assumptions* (H1)*, there exists a real number $\delta_0 > 0$ and a real number $k \in]0, 1[$ such that $\mathcal{K}_n^K$ has a unique fixed point x_n^K in $\mathcal{B}(x^*, \delta_0)$ and*

$$\frac{\alpha_n}{1+k} \leqslant \left\|x_n^K - x^*\right\| \leqslant \frac{\alpha_n}{1-k}, \tag{9}$$

where

$$\alpha_n = \left\|\left(I - D\mathcal{K}_n^K(x^*)\right)^{-1}\left[x^* - \mathcal{K}_n^K(x^*)\right]\right\|.$$

Also

$$\alpha_n \leqslant 2c\left\{(1 + p)r_n + a_n\right\}\|P_n x^* - x^*\|, \tag{10}$$

so that $\|x_n^K - x^*\|/\|P_n x^* - x^*\| \to 0$.

Proof. The proof is inspired from the results of [7] and [3]. The main idea is to prove that $\mathcal{K}_n^K$ is a contraction in a neighbourhood of x^*. Let $A_n = I - \mathcal{K}_n^K$. Since by Lemma 2.2, $DA_n(x^*) = I - D\mathcal{K}_n^K(x^*)$ is invertible, it follows that $A_n x = 0$ is equivalent to $x = B_n x$ where

$$B_n x = x^* - DA_n(x^*)^{-1}\big\{A_n x^* + A_n x - A_n x^* - DA_n(x^*)(x - x^*)\big\}.$$

Hence

$$\|B_n x - x^*\| \leqslant \big\|DA_n(x^*)^{-1}\{A_n x^*\}\big\| + \big\|DA_n(x^*)^{-1}\big\{A_n x - A_n x^* - DA_n(x^*)(x - x^*)\big\}\big\|.$$

Since $A_n x^* = (P_n - I)[\mathcal{K}(P_n x^*) - \mathcal{K}(x^*) - L(P_n x^* - x^*)] + (P_n - I)L(P_n x^* - x^*)$, by Lemma 2.2,

$$\big\|DA_n(x^*)^{-1}\{A_n x^*\}\big\| \leqslant 2c\big\{(1+p)r_n + a_n\big\}\big\|P_n x^* - x^*\big\|. \tag{11}$$

By the generalized mean value theorem applied to $A_n - DA_n(x^*)$ and using Lemma 2.3, we obtain

$$\begin{aligned}\big\|A_n x - A_n x^* - DA_n(x^*)(x - x^*)\big\| &\leqslant \|x - x^*\| \sup_{0<\theta<1} \big\|D\mathcal{K}_n^K(x^*) - D\mathcal{K}_n^K\big(x^* + \theta(x - x^*)\big)\big\| \\ &\leqslant 2cm\|x - x^*\|^2.\end{aligned}$$

Hence for $x \in B(x^*, \delta)$, $\|B_n x - x^*\| \leqslant 2c\delta_n\|P_n x^* - x^*\| + 2cm\|x - x^*\|^2$, where $\delta_n = (1+p)r_n + a_n$.

Let k be real number, $0 < k < 1$ and $\delta_0 < \delta$ be such that $2cm\delta_0 < k$.

Let n_1 be an integer such that $n \geqslant n_1$, $2c\delta_n\|P_n x^* - x^*\| \leqslant \delta_0(1-k)$.

Then for $n \geqslant n_1$ and for x in $B(x^*, \delta_0)$, $\|B_n x - x^*\| \leqslant \delta_0(1-k) + k\delta_0 = \delta_0$, which proves that B_n maps $B(x^*, \delta_0)$ into itself. In a similar manner, it can be shown that B_n is a contraction on $B(x^*, \delta_0)$. Hence by the contraction mapping principle, B_n has a unique fixed point x_n^K in $B(x^*, \delta_0)$.

We have $x_n^K - x^* = B_n x_n^K - x^* = -DA_n(x^*)^{-1}\{A_n x^*\} - DA_n(x^*)^{-1}\{A_n x_n^K - A_n x^* - DA_n(x^*)(x_n^K - x^*)\}$, so that

$$\big\|DA_n(x^*)^{-1}\{A_n x^*\}\big\| - 2cm\big\|x_n^K - x^*\big\|^2 \leqslant \big\|x_n^K - x^*\big\| \leqslant \big\|DA_n(x^*)^{-1}\{A_n x^*\}\big\| + 2cm\big\|x_n^K - x^*\big\|^2.$$

Thus, as $2cm\|x_n^K - x^*\| \leqslant 2cm\delta_0 < k$, we obtain

$$\frac{\alpha_n}{1+k} \leqslant \big\|x_n^K - x^*\big\| \leqslant \frac{\alpha_n}{1-k}. \qquad \square$$

Remark 2. Note that the rate of convergence of $\|x_n^K - x^*\|$ is determined by α_n and from (11),

$$\frac{\alpha_n}{\|P_n x^* - x^*\|} \leqslant 2c\big\{(1+p)r_n + a_n\big\} \to 0.$$

Hence

$$\frac{\|x_n^K - x^*\|}{\|P_n x^* - x^*\|} \to 0$$

under the assumptions (H_1). As a consequence, x_n^K converges to x^* faster than x_n^G or x_n^S.

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