Abstract

We prove that a singular foliation on a compact manifold admitting an adapted Riemannian metric for which all leaves are minimal must be regular. To cite this article: V. Miquel, R.A. Wolak, C. R. Acad. Sci. Paris, Ser. I 342 (2006).
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Résumé

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1. Introduction

Let $M$ be a smooth manifold. A generalized differentiable distribution $\mathcal{D} \subset TM$ (i.e. $\dim \mathcal{D} \times x \in M$ is not constant) is called a foliation if it is completely integrable, i.e. if at any point $x$ of $M$ it admits a maximal integral submanifold, called a leaf $s$ of the foliation $\mathcal{D}$. The Sussmann–Stefan–Frobenius theorem provides necessary and sufficient conditions for complete integrability of generalized differentiable distributions, cf. [6]. A foliation $\mathcal{F}$ on a smooth manifold $M$ (cf. [3], p. 189) is a Riemannian foliation (RF for short) if there is a Riemannian metric on $M$ adapted to $\mathcal{F}$ in the sense that every geodesic which is orthogonal to a leaf at one point remains orthogonal to every leaf it meets. A Riemannian foliation $\mathcal{F}$ is called regular (RRF) if all the leaves have the same dimension. Otherwise, it is called singular (SRF). In an SRF, the leaves of maximal dimension are called regular, and the others are called singular leaves.

A foliation $\mathcal{F}$ on a manifold $M$ is taut if there is a metric $g$ on $M$ such that every leaf of $\mathcal{F}$ is minimal. The study of these foliations was stimulated by Haefliger’s paper cf. [1] in which the author demonstrated that in the regular case ‘tautness’ is a transverse property. In 1983, Carrière proposed the following conjecture for compact manifolds: A RRF is taut if and only if the basic cohomology $H^{\dim \mathcal{F}}(M/\mathcal{F}) \neq 0$. The problem was finally solved by Masa,
2. Preliminaries

Given a SRF $\mathcal{F}$ on $M$, we shall fix an adapted Riemannian metric $(\cdot, \cdot)$ on $M$. If $x \in M$, then $L_x$ is the leaf of $\mathcal{F}$ passing through the point $x$. Given any leaf $L$ of $\mathcal{F}$ and a connected open set $P$ in $L$, we shall denote by $P_r$ (resp. $\partial P_r$) the tube (resp. the tubular hypersurface) of radius $r$ centered at $P$, that is

$$ P_r = \{\exp tu; u \in N_x P, x \in P, 0 \leq t < r\}, \quad \partial P_r = \{\exp u; u \in N_x P, x \in P\}, $$

where $N_x P$ denotes the unit sphere fiber at $x$ of the normal bundle of $P$, and $\exp$ denotes the exponential map for the metric $(\cdot, \cdot)$. Crucial for our result is the following structure theorem of a SRF:

**Theorem 2.1** (Homothety Lemma, [3], p. 193). Given $x_0 \in M$ compact, let $P$ be a relatively compact connected open neighborhood of $x_0$ in the leaf $L_{x_0}$ through $x_0$. Then, there is a $\rho_0 > 0$ such that,

(i) for every $x \in P_{\rho_0}$, the connected component $P_x$ of $x$ in $L_x \cap P_{\rho_0}$ (called a plaque of $F$ through $x$) is contained in $\partial P_{\text{dist}(P, x)}$, and

(ii) for $\lambda > 0$ and $\rho > 0$ such that $\rho$ and $\lambda \rho$ are both $< \rho_0$, the diffeomorphism $\partial P_\rho \rightarrow \partial P_{\lambda \rho}$ defined by $\exp \rho u \mapsto \exp \lambda \rho u$ sends one plaque onto one plaque.

Moreover, we shall use the well known first variation formula for the volume of a submanifold $R$ of $M$ when we take a normal variation $R_t = \{\exp y, t X_y; y \in R\}$ of $R$ ($X$ being a vector field normal to $R$ called variation vector field)

$$ \frac{d}{dt} \text{volume}(R_t)_{t=0} = \int_R \langle H_y, X_y \rangle \eta, $$

where $\eta$ is the volume element of $R$ and $H$ is the mean curvature vector of $R$.

3. Proof of Theorem 1.1

The idea of the proof is the following: If $\mathcal{F}$ is a SRF, the Homothety Lemma (Theorem 2.1) implies that, in a suitable neighborhood of an open set of a singular leaf $L$ of dimension $q$, the regular leaves $R$ of dimension $q + k$ look like tubular submanifolds around $L$, with spherical shaped slices of dimension $k$ (the vectors $Z_{q+j}(s)$ in the formula (4) below are tangent to these slices). Then, in the direction from $R$ to $L$, and near $L$, the contribution of these spherical shaped slices makes the volume of the regular leave to decrease (this is the meaning of formula (12) below). Since the mean curvature gives the variation of the volume of the regular leaves (formula (2)), it cannot be zero. This incompatibility with minimality disappears only when there are no spherical shaped slices, that is, when $k = 0$. Now, let us go into the details.

Let us suppose that $\mathcal{F}$ is a SRF. Let $q + k$ be the maximal dimension of the leaves of the foliation, and $q$ the dimension of a singular leaf $L$ such that, being $S$ a relatively compact connected open set in $L$, in the tube $S_{\rho_0}$, with the $\rho_0$ given in Theorem 2.1, there is a plaque $R$ of a leaf of maximal dimension. From Theorem 2.1(i) it follows that $R$ is at constant distance $r$ from $S$ and there is a subset $\mathcal{U}$ of the unit normal bundle $\mathcal{N}S$ of $S$ such that $R = \exp(r\mathcal{U})$.

Moreover, it is a consequence of Theorem 2.1(ii) that, for any $s \in (0, r)$, $R_s = \exp(s\mathcal{U})$ is a plaque of a leaf of dimension $q + k$ of the foliation. The subset $\mathcal{U}$ can be written as

$$ \mathcal{U} = \bigcup_{x \in S} U_x, \quad U_x = \mathcal{U} \cap N_x S. $$
Let us consider on $\mathcal{U}$ the volume form $dx \wedge du_x$, where $dx$ is the volume form of $S$ and $du_x$ is the volume form of $U_x$. As $R_s = \exp(s\mathcal{U})$, we have the following diffeomorphism:

$$\varphi: \mathcal{U} \subset N S \longrightarrow R_s \subset M \quad \text{defined by} \quad U_x \ni u_x \mapsto \exp u_x.$$

Then the volume form $\eta_s$ of $R_s$ has the pullback $\varphi^*\eta_s$ that can be written as

$$\varphi^*\eta_s = \phi(s, x, u_x) dx \wedge du_x,$$

where $\phi$ can be computed, taking orthonormal basis $\{e_1, \ldots, e_k\}$ on $T_x S$ and $\{e_{q+1}, \ldots, e_{q+k}\}$ of $T_{u_x} U_x$ (recall that $k = \dim U_x = \dim R_s - \dim S$) as follows

$$\begin{align*}
\phi(s, x, u_x) &= \varphi^*(\eta_s(e_1, \ldots, e_q, e_{q+1}, \ldots, e_{q+k}) \\
&= \eta_s(\exp_{u_x} se_1, \ldots, \exp_{u_x} se_q, \exp_{u_x} se_{q+1}, \ldots, \exp_{u_x} se_{q+k}) \\
&= \eta_s(Y_1(s), \ldots, Y_q(s), Z_{q+1}(s), \ldots, Z_{q+k}(s)),
\end{align*}$$

(4)

where $Y_j(t), Z_{q+j}(t)$ (see, for instance [4], pages 36 and 58–60) are the Jacobi fields along the geodesic $t \mapsto \exp_t u_x$, $0 \leq t \leq r$, satisfying

$$Y_j(0) = e_i, \quad Y'_{j}(0) + L_{u_x} e_i \in (T_s S)\perp,$$

(5)

where $L_{u_x}$ is the Weingarten map of $S$ in the direction of $u_x$.

$$Z_{q+j}(0) = 0, \quad Z'_{q+j}(0) = e_{q+j}, \quad \text{and}$$

(6)

$$\lim_{t \to 0} \frac{Z_{q+j}(t)}{t} = \lim_{t \to 0} \frac{1}{t} \exp_{u_x} (e_{q+j}) = e_{q+j}.$$

(7)

On the other hand, it follows from (3) that volume($R_s$) = $\int_{R_s} \eta_s = \int_{S} \int_{U_x} \phi(s, x, u_x) dx \wedge du_x$, and

$$\frac{d}{ds} \text{volume}(R_s) = \int_{S} \int_{U_x} \frac{d}{ds} \phi(s, x, u_x) dx \wedge du_x = \int_{S} \int_{U_x} \frac{d}{ds} \phi(s, x, u_x) \phi(s, x, u_x) dx \wedge du_x$$

$$= \int_{R_s} \frac{d}{ds} \phi(s, x, u_x) \eta_s.$$

(8)

On the other hand, when we apply formula (2) to the submanifold $R_s$ and the variation vector field $\exp_s u_x \mapsto \frac{d}{ds} (\exp_s u_x)$, we obtain

$$\frac{d}{ds} \text{volume}(R_s) |_{s=0} = \int_{R_s} H_{\exp_s u_x} \cdot \frac{d}{ds} (\exp_s u_x) \eta_s,$$

(9)

$H_{\exp_s u_x}$ being the mean curvature of $R_s$ at $\exp_s u_x$. Now, let $f : R_s \rightarrow \mathbb{R}$ be the $C^\infty$ function

$$f(\exp_s u_x) = \phi(s, x, u_x)^{-1} \frac{d}{ds} \phi(s, x, u_x) - \left( H_{\exp_s u_x} \cdot \frac{d}{ds} (\exp_s u_x) \right).$$

Let $R^+_s$ (resp. $R^-_s$) be the set of points of $R_s$ where $f$ is $>0$ (resp. $<0$). The same arguments used to prove formulae (8) and (9) show that they are still true for $R^+_s$ and $R^-_s$. Then

$$\int_{R^+_s} f(\exp_s u_x) \eta_s = 0 = \int_{R^-_s} f(\exp_s u_x) \eta_s,$$

and these equalities imply that $R^+_s = \emptyset = R^-_s$. Then, $f$ is identically zero, that is, for every $\exp_s u_x \in R_s$,

$$\phi(s, x, u_x)^{-1} \frac{d}{ds} \phi(s, x, u_x) = \left( H_{\exp_s u_x} \cdot \frac{d}{ds} (\exp_s u_x) \right).$$

(10)
Let $\xi_1, \ldots, \xi_l$, $l = n - q - k$, be a family of unit orthogonal vector fields along $\exp_x t u_x$ which are orthogonal to $Y_1, \ldots, Z_{q+k}$. Then, for every $t \in [0, s]$,

$$
\eta_s(Y_1, \ldots, Y_q, Z_{q+1}, \ldots, Z_{q+k}) = \omega(Y_1, \ldots, Y_q, Z_{q+1}, \ldots, Z_{q+k}, \xi_1, \ldots, \xi_l),
$$

where $\omega$ is the volume form of $M$. Then

$$
\frac{d}{ds} \phi(s, x, u_x) = \sum_i \omega(Y_1, \ldots, Y'_i, \ldots, Y_q, Z_{q+1}, \ldots, \xi_l)
$$

$$
+ \sum_j \omega(Y_1, \ldots, Y_q, Z_{q+1}, \ldots, Z'_{q+j}, \ldots, Z_{q+k}, \xi_1, \ldots, \xi_l)
$$

$$
+ \sum_{\alpha} \omega(Y_1, \ldots, Z_{q+k}, \xi_1, \ldots, \xi'_\alpha, \ldots, \xi_l).
$$

Since $\langle \xi_\alpha, \xi_\alpha \rangle = 1$, the vector fields $\xi'_\alpha$ and $\xi_s$ are orthogonal. Therefore, the last term of the sum vanishes. Using this, we can write

$$
\frac{d\phi}{ds} s^{-k-1} = \sum_i \omega(Y_1, \ldots, Y'_i, \ldots, Y_q, \frac{Z_{q+1}}{s}, \ldots, \frac{Z_{q+k}}{s}, \xi_1, \ldots, \xi_l)
$$

$$
+ \sum_j \omega(Y_1, \ldots, Y_q, \frac{Z_{q+1}}{s}, \ldots, \frac{Z_{q+j}}{s}, \ldots, \frac{Z_{q+k}}{s}, \xi_1, \ldots, \xi_l).
$$

From (6), $Z_{q+1}(0) = 0$, then the first adding term in (11) goes to 0 as $s$ tends to 0 (since the other vectors remain bounded). From (5), (6) and (7) the second adding term in (11) has limit 1 when $s$ goes to 0, then

$$
\lim_{s \to 0} s^{-(k-1)} \frac{d\phi}{ds} = 1,
$$

and therefore, near $s = 0$, $\frac{d}{ds} \phi(s, x, u_x) \neq 0$. Thus, it follows from (10) that the mean curvature of this $R_s$ is not identically 0.

We have started with a SRF $F$ on $M$, fixed an arbitrary adapted Riemannian metric $\langle \cdot, \cdot \rangle$, picked an arbitrary singular leaf of dimension $q < q + k$ and showed that, in a neighborhood of it there are regular leaves whose mean curvature is not identically 0. So, the proof of Theorem 1.1 is finished.

References


