Partial Differential Equations

Two new discrete inequalities of Poincaré–Friedrichs on discontinuous spaces for Maxwell’s equations

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Abstract

We present two new discrete inequalities of Poincaré–Friedrichs on discontinuous spaces for Maxwell’s equations. The proofs of the inequalities are based on some decompositions formulas of \( L^2(\Omega)^3 \).

Résumé

Deux nouvelles inégalités de type Poincaré–Friedrichs sur les espaces discontinus pour les équations de Maxwell. On présente deux nouvelles inégalités de type Poincaré–Friedrichs sur les espaces discontinus. La preuve des inégalités est basée sur des formules de décomposition orthogonale de \( L^2(\Omega)^3 \).

1. Some notations and spaces

Throughout this Note, \( \Omega \) will denote a bounded Lipschitz polyhedron included in \( \mathbb{R}^3 \) which is supposed to be both connected and simply connected. \( \Gamma \) is the boundary of \( \Omega \) which is also assumed to be connected and simply connected. Given a domain \( D \) in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \), we denote by \( H^s(D)^d, \) \( d = 1, 2, 3, \) the Sobolev space of real valued functions with integer or fractional regularity exponent \( s \geq 0 \), endowed with the norm \( \| \cdot \|_{s,D} \); see, e.g., [3]. For \( D \subset \mathbb{R}^3 \), \( H(\nabla \times ,D) \) and \( H(\nabla \cdot ,D) \) are the spaces of real valued vector functions \( u \in L^2(D)^3 \) with \( \nabla \times u \in L^2(D)^3 \) and \( \nabla \cdot u \in L^2(D) \), respectively, endowed with the graph norms. We denote by \( H^1_0(D), H_0(\nabla \times ,D), H_0(\nabla \cdot ,D) \) the subspaces of \( H^1(D), H(\nabla \times ,D), H(\nabla \cdot ,D) \) of functions with zero trace, tangential trace and normal trace on \( \partial D \), respectively. The spaces \( H(\nabla \times 0,D) \) and \( H(\nabla \cdot 0,D) \) are the subspaces of \( H(\nabla \times ,D) \) and \( H(\nabla \cdot ,D) \) consisting of irrotational and divergence-free functions, respectively. We assume that \( \Omega \) satisfies \( H_0(\nabla \times ,\Omega) \cap H(\nabla \cdot ,\Omega) \) and \( H(\nabla \times ,\Omega) \cap H_0(\nabla \cdot ,\Omega) \) are both continuously imbedded in \( H^1(\Omega)^3 \). Let \( \Pi_h \) be a partition into tetrahedra for \( \Omega \). If \( K \) in \( \Pi_h \) we denote by \( h_K \) the diameter of \( K \) and set \( h := \max_{K \in \Pi_h} h_K \).

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Faces. We define and characterise the faces of the triangulation $\Pi_h$. An interior face of $\Pi_h$ is defined as the (non-empty) two-dimensional interior of $\partial K_1 \cap \partial K_2$, where $K_1$ and $K_2$ are two adjacent elements of $\Pi_h$. A boundary face of $\Pi_h$ is defined as the (non-empty) two-dimensional interior of $\partial K \cap \partial \Omega$, where $K$ is a boundary element of $\Pi_h$. We denote by $F_h^I$ the union of all interior faces of $\Pi_h$, by $F_h^D$ the union of all boundary faces of $\Pi_h$ and let $F_h$ denote the union of all faces of $\Pi_h$. Furthermore we associate $F_h^D$ with $\Gamma$ since $\Omega$ is a polyhedron.

Traces. Let $H^s(\Pi_h) = \{v; v|_K \in H^1(K) \forall K \in \Pi_h\}$ for $s > \frac{1}{2}$ be endowed with the norm $\|v\|_{s,\Pi_h} = \sum_{K \in \Pi_h} \|v\|_{s,K}^2$. Then the elementwise traces of functions in $H^s(\Pi_h)$ belong to the space $\text{TR}(F_h) := \Pi_{K \in \Pi_h} L^2(\partial K)$; they are double-valued on $F_h^I$ and single-valued on $F_h^D$. The space $L^2(F_h)$ can be identified with the functions in $\text{TR}(F_h)$ for which the two traces values coincide.

Trace operators. Let us introduce the following trace operators for piecewise smooth functions. First, let $w \in \text{TR}(F_h)^3$ and $e \subset F_h$. If $e$ is an interior face in $F_h^I$, we denote by $K_1$ and $K_2$ the elements sharing $e$, by $n_i$ the normal unit vector pointing exterior to $K_i$ and we set $\omega_i = \omega|_{\partial K_i}$, $i = 1, 2$. We define the average, tangential and normal jumps of $w$ at $x \in e$ as

$$\{\omega\} = \frac{\omega_1 + \omega_2}{2}, \quad [\omega]_T = n_1 \times \omega_1 + n_2 \times \omega_2 \quad \text{and} \quad [\omega]_N = n_1 \cdot \omega_1 + n_2 \cdot \omega_2.$$  

If $e \subset F_h^D$, we set for $x \in e$

$$\{\omega\} = \omega, \quad [\omega]_T = n \times \omega \quad \text{and} \quad [\omega]_N = n \cdot \omega.$$  

We denote by $(\cdot, \cdot)$ the scalar product in $L^2(\Omega)^3$ or $L^2(\Omega)$ and by $\|\cdot\|_0,\Omega = \|\cdot\|_{L^2(\Omega)}$ or $\|\cdot\|^3_{L^2(\Omega)}$. For $e \subset F_h$, we denote by $(\cdot, \cdot)_e$ the scalar product in $L^2(\mathbb{e})^3$ or $L^2(\mathbb{e})$. Furthermore if $F_h^D$ is identified to $\partial \Omega$, we identify $\sum_{e \subset F_h^D} (\cdot, \cdot)_e$ to $(\cdot, \cdot)$, the scalar product in $L^2(\partial \Omega)^3$ or $L^2(\partial \Omega)$. In the previous notation we can state the basic integration by parts formulas

$$\forall v, u \in H^1(\Pi_h)^3, \forall \psi \in H^1(\Pi_h), \text{we have}$$

$$(\nabla \times u, v) = (u, \nabla \times v) + \langle n \times u, v \rangle + \sum_{e \subset F_h^I} \left( [u]_T, \{v\} \right)_e - \left( [v]_T, \{u\} \right)_e \quad (1)$$

and

$$(\nabla \cdot u, \psi) = (u, \nabla \psi) + \langle u \cdot n, \psi \rangle + \sum_{e \subset F_h^D} \left( [u]_N, \{\psi\} \right)_e + \left( [\psi], \{u\} \cdot n \right)_e. \quad (2)$$

2. The first inequality

Lemma 2.1. Let $u \in H^1(\Pi_h)^3$ and let $\sigma = \frac{1}{h}$. Then, there exists a constant $C$ independent of $h$ such that

$$\|u\|^2 \leq C \left( \|\nabla \times u\|^2 + \|\nabla \cdot u\|^2 + \sum_{e \subset F_h^I} \|\sqrt{\sigma} [u]_T\|^2_{0,e} + \sum_{e \subset F_h^D} \|\sqrt{\sigma} [u]_N\|^2_{0,e} \right).$$

Proof. Let us first denote that the following orthogonal decomposition formula holds if $\partial \Omega$ is simply-connected (see [1])

$$L^2(\Omega)^3 = H_0(\nabla \times 0, \Omega) \oplus H(\nabla \cdot 0, \Omega).$$

Now, let $u \in H^1(\Pi_h)^3$, then $u \in L^2(\Omega)^3$ and we can decompose $u$ as

$$u = u_1 + u_2 \quad \text{with} \quad u_1 \in H_0(\nabla \times 0, \Omega) \text{ and } u_2 \in H(\nabla \cdot 0, \Omega). \quad (3)$$

As in [1], we show that $u_1 \in H_0(\nabla \times 0, \Omega)$ if and only if $u_1 = \nabla q$ with $q \in H^1_0(\Omega)$. We also show that $u_2 = \nabla \phi$ with $\phi \in H(\nabla \times, \Omega) \cap H_0(\nabla \cdot 0, \Omega)$. In particular, the traces of $\phi$ are well defined since $\phi \in H_0(\nabla \times, \Omega) \cap H(\nabla \cdot, \Omega) \leftrightarrow$
Now, by using trace inequality (see [4]), we have for any $e$
\[\| u \|^2 = (\nabla q + \nabla \times \phi, \nabla q + \nabla \times \phi) = (\nabla q, \nabla q) + (\nabla \times \phi, \nabla \times \phi) = \| \nabla q \|^2 + \| \nabla \times \phi \|^2.\]

Then, since $q$ is in $H^1_0(\Omega)$,
\[\| u \|^2 = -(\nabla \cdot u, q) + (\nabla \times u, \phi) + \sum_{e \subset F_h} \left( \left[ [u]_N, q \right]_e - \left[ [u]_T, \phi \right]_e \right) + \sum_{e \subset F_h^0} \left( \left[ n \times u, \phi \right]_e \right).
\]

So
\[\| u \|^2 \leq C \left( \| \nabla \cdot u \|^2 + \| \nabla \times u \|^2 + \sum_{e \subset F_h^I} \| \sqrt{\sigma} [u]_N \|^2_{0,e} + \sum_{e \subset F_h} \| \sqrt{\sigma} [u]_T \|^2_{0,e} \right)^{1/2}
\times \left( \| q \|^2 + \| \phi \|^2 + \sum_{e \subset F_h^I} \left\| \frac{1}{\sqrt{\sigma}} q \right\|^2_{0,e} + \sum_{e \subset F_h} \left\| \frac{1}{\sqrt{\sigma}} \phi \right\|^2_{0,e} \right)^{1/2}.
\]

It is clear that
\[\| q \|^2 \leq C(\Omega) \| \nabla q \|^2 \leq C(\Omega) \| u \|^2.
\]

Since $\phi \in H(\nabla \times, \Omega) \cap H_0(\nabla \cdot, \Omega)$ and $\nabla \cdot \phi = 0$, we obtain (see [1] for the first inequality)
\[\| \phi \|^2 \leq C(\Omega) (\| \nabla \times \phi \|^2 + \| \nabla \cdot \phi \|^2)
\leq C(\Omega) \| \nabla \times \phi \|^2
\leq C(\Omega) \| u \|^2.
\]

Now, by using trace inequality (see [4]), we have for any $e \subset F_h$
\[\left\| \frac{1}{\sqrt{\sigma}} q \right\|^2_{0,e} \leq \frac{C}{\sigma} \left( \frac{1}{h_K} \| q \|^2_{\partial K} + \| q \|_{0,K} \| \nabla q \|_{0,K} \right)
\leq Ch \left( \frac{1}{h_K} \| q \|^2_{\partial K} + \frac{1}{h_K} \| q \|^2_{0,K} + h_K \| \nabla q \|^2_{0,K} \right)
\leq Ch \left( \frac{1}{h_K} \| q \|^2_{\partial K} + \frac{1}{h_K} \| q \|^2_{0,K} + \frac{1}{h_K} \| \nabla q \|^2_{0,K} \right)
\leq C(\| q \|^2_{0,K} + \| \nabla q \|^2_{0,K}).
\]

In particular
\[\sum_{e \subset F_h^I} \left\| \frac{1}{\sqrt{\sigma}} q \right\|^2_{0,e} \leq C \sum_{K \in \Pi_h} (\| q \|^2_{0,K} + \| \nabla q \|^2_{0,K})
\leq C(\| q \|^2 + \| \nabla q \|^2)
\leq C(\| u \|^2).
\]

In the same manner, using the imbedding of $H(\nabla \times, \Omega) \cap H_0(\nabla \cdot, \Omega)$ in $H^1(\Omega)^3$; we can bound $\sum_{e \subset F_h} \left\| \frac{1}{\sqrt{\sigma}} \phi \right\|_{0,e}^2$ and obtain
\[ \left\| \frac{1}{\sqrt{\sigma}} \phi \right\|_{0,F_h}^2 \leq C \| \phi \|_{1,\Omega}^2 \leq C \| \phi \|_{H(\nabla \times \cdot, \Omega) \cap H_0(\nabla \cdot \cdot, \Omega)} \leq C \left( \| \nabla \times \phi \|^2 + \| \nabla \cdot \phi \| \right) \]
\[
\leq C \| \nabla \times \phi \|^2 \leq C \| u \|^2.
\]
Finally, we obtain
\[
\| u \|^2 \leq C \left( \| \nabla \cdot \phi \|^2 + \| \nabla \times \phi \|^2 + \sum_{e \subset F_h^I} \sqrt{\sigma} [u]_N^2_0,e + \sum_{e \subset F_h} \sqrt{\sigma} [u]_T^2_0,e \right)^{1/2} \| u \|,
\]
which is equivalent to
\[
\| u \|^2 \leq C \left( \| \nabla \cdot \phi \|^2 + \| \nabla \times \phi \|^2 + \sum_{e \subset F_h^I} \sqrt{\sigma} [u]_N^2_0,e + \sum_{e \subset F_h} \sqrt{\sigma} [u]_T^2_0,e \right).
\]

3. The second inequality

Lemma 3.1. Let \( u \in H^1(\Pi_h)^3 \) and let \( \sigma = \frac{1}{h} \). Then, there exists \( C \) independent of \( h \) such that
\[
\| u \|^2 \leq C \left( \| \nabla \cdot \phi \|^2 + \| \nabla \times \phi \|^2 + \sum_{e \subset F_h^I} \sqrt{\sigma} [u]_N^2_0,e + \sum_{e \subset F_h} \sqrt{\sigma} [u]_T^2_0,e \right).
\]

Proof. The proof is similar to the proof in the previous section. But here we use the following orthogonal decomposition formula if \( \Omega \) is simply-connected (see also [1,2])

\[
L^2(\Omega)^3 = H(\nabla \times 0, \Omega) \oplus H_0(\nabla \cdot 0, \Omega).
\]

Then, for \( u \in L^2(\Omega)^3 \) we write
\[
u = u_1 + u_2
\]
with \( u_1 \in H(\nabla \times 0, \Omega) \) and \( u_2 \in H_0(\nabla \cdot 0, \Omega) \). Since \( \nabla \times u_1 = 0 \), we write \( u_1 = \nabla q \) with \( q \in H^1(\Omega) \) and since \( u_2 \in H_0(\nabla \cdot 0, \Omega) \), we write \( u_2 = \nabla \times \varphi \) with \( \varphi \in H_0(\nabla \times \cdot, \Omega) \cap H(\nabla \cdot 0, \Omega) \) (see [1,2]).

References