Abstract

We prove that every Cantor aperiodic system is homeomorphic to the Vershik map acting on the space of infinite paths of an ordered Bratteli diagram and give several corollaries of this result. To cite this article: K. Medynets, C. R. Acad. Sci. Paris, Ser. I 342 (2006).

Résumé


1. Introduction

Every Cantor minimal system can be represented as the Vershik map acting on an ordered Bratteli diagram [7]. This representation turns out to be a powerful tool in the study of orbit equivalence of Cantor minimal systems [4–6,8]. The goal of this Note is to find a similar realization of every aperiodic homeomorphism of a Cantor set and give several applications of this result.

We say that \((X, T)\) is a Cantor aperiodic system if \(X\) is a Cantor set and \(T\) is a homeomorphism of \(X\) without periodic points. By a \(T\)-tower \(\xi\), we mean a family of disjoint clopen sets \(\{B, TB, \ldots, T^{n-1}B\}\). The set \(B\) is called the base of \(\xi\) and denoted by \(B(\xi)\); the number \(n\) is called the height of \(\xi\) and denoted by \(h(\xi)\). A clopen partition \(\mathcal{E} = \{\xi_1, \ldots, \xi_m\}\) of \(X\) is called a Kakutani–Rokhlin (K–R) partition if all the \(\xi_i\)’s are disjoint \(T\)-towers. The sets \(\{T^iB(\xi) : \xi \in \mathcal{E}, i = 1, \ldots, h(\xi) - 1\}\) are called atoms of \(\mathcal{E}\). We say that a K–R partition \(\mathcal{E}_2\) refines a K–R partition \(\mathcal{E}_1\) if every atom of \(\mathcal{E}_1\) is a union of some atoms of \(\mathcal{E}_2\). For a K–R partition \(\mathcal{E} = \{\xi_1, \ldots, \xi_n\}\), set \(B(\mathcal{E}) = \bigcup_{1 \leq i \leq n} B(\xi_i)\).

Let \((X, T)\) be a Cantor aperiodic system and let \(A\) be a clopen set. We say that \(A\) is a complete \(T\)-section if \(A\) meets every \(T\)-orbit at least once; a point \(x \in A\) is called recurrent with respect to \(A\) if there exists \(n \in \mathbb{N}\) such that \(T^n x \in A\). If a clopen complete \(T\)-section \(A\) consists of recurrent points, then the function of first return \(n_A(x) = \min\{n \in \mathbb{N} : T^n x \in A\}\) determines a K–R partition \(\mathcal{E}\) of \(X\) such that \(B(\mathcal{E}) = A\).
2. Bratteli diagrams

In this section, we show that every Cantor aperiodic system is homeomorphic to the Vershik map acting on the space of infinite paths of an ordered Bratteli diagram.

**Theorem 2.1.** Let $(X, T)$ be a Cantor aperiodic system. There exists a sequence of $K$–R partitions $\{\Xi_n\}$ of $X$ such that for all $n \geq 1$: (i) $\Xi_{n+1}$ refines $\Xi_n$; (ii) $h_{n+1} > h_n$, where $h_n = \min \{h(\xi) : \xi \in \Xi_n\}$; (iii) $B(\Xi_n) \supset B(\Xi_{n+1})$; (iv) the sequence $\{\Xi_n\}$ generates the clopen topology of $X$.

The proof is obtained by consequent application of the following lemma. Note that part (ii) was originally proved in [1].

**Lemma 2.2.** Let $(X, T)$ be a Cantor aperiodic system.

(i) If $A$ is a clopen complete $T$-section, then $A$ consists of recurrent points.

(ii) For every $n > 0$, there exists a clopen $K$–R partition $\Xi = \{\xi_1, \ldots, \xi_{k_n}\}$ of $X$ such that the height of every $T$-tower $h(\xi_i)$ is at least $n$.

**Proof.** Since statement (ii) is principal, we sketch its proof. For every $x \in X$, find a clopen neighborhood $U_x$ such that $T^i U_x \cap U_x = \emptyset$ for $i = 1, \ldots, n - 1$. Choose a finite subcover $X = U_1 \cup \cdots \cup U_k$, where $U_i = U_{x_i}$, $i = 1, \ldots, k$.

Set $A_1 = U_1$ and $A_m = U_m - \bigcup_{j=-m-1}^{m-1} T^j (A_1 \cup \cdots \cup A_{m-1})$, $m = 2, \ldots, k$. Notice that $A = A_1 \cup \cdots \cup A_k$ is a clopen complete $T$-section. By (i), the set $A$ consists of recurrent points. To find $\Xi$, we apply the function of first return to $A$. □

Let $(X, T)$ be a Cantor aperiodic system; we say that a closed set $Y \subset X$ is a basic set if every clopen neighborhood of $Y$ is a complete $T$-section and $Y$ meets every $T$-orbit at most once. If a sequence of $K$–R partitions $\{\Xi_n\}$ satisfies the conditions of Theorem 2.1, then $Y = \bigcap_n B(\Xi_n)$ is a basic set. Thus,

**Corollary 2.3.** Every Cantor aperiodic system has a basic set.

For the notions related to ordered Bratteli diagrams we refer the reader to [4,7]. We recall only the definition of the Vershik map: let $B = (V, E, \geq)$ be an ordered Bratteli diagram with the path space $X_B$ and let $X_{\max}, X_{\min}$ be the sets of all maximal and minimal paths, respectively. We say that a homeomorphism $\varphi_B : X_B \to X_B$ is a Vershik map if $\varphi_B(X_{\max}) = X_{\min}$ and if $x = (x_1, x_2, \ldots, x_k) \neq X_{\max}$, then $\varphi(x_1, x_2, \ldots) = (x_1', \ldots, x_k', x_{k+1}, x_{k+2}, \ldots)$, where $k = \min \{n \geq 1 : x_n$ is not maximal], $x_k'$ is the successor of $x_k$, and $(x_1', \ldots, x_{k-1}')$ is the minimal path connecting the top vertex $v_0$ with the source of $x_k$. We call the system $(X_B, \varphi_B)$ a Bratteli–Vershik model.

Let $Y$ be a basic set for a Cantor aperiodic system $(X, T)$. Take a sequence of $K$–R partitions $\{\Xi_n\}$ satisfying the conditions of Theorem 2.1 such that $Y = \bigcap_n B(\Xi_n)$. Applying the method used in [7, Section 4] to $\{\Xi_n\}$, we prove:

**Theorem 2.4.** Let $(X, T, Y)$ be a Cantor aperiodic system with a basic set $Y$. There exists an ordered Bratteli diagram $B = (V, E, \geq)$ such that $(X, T)$ is homeomorphic to a Bratteli–Vershik model $(X_B, \varphi_B)$ and the homeomorphism complementing the conjugacy between $T$ and $\varphi_B$ maps the basic set $Y$ onto the set of all minimal paths of $X_B$. The equivalence class (generated by the isomorphism and telescoping) of the diagram $B$ does not depend on a choice of $\{\Xi_n\}$ with $\bigcap_n B(\Xi_n) = Y$.

It is interesting to describe the variety of Bratteli diagrams corresponding to Cantor aperiodic systems. In other words, the question is whether a given Bratteli diagram has an ordering such that the Vershik map is a well-defined aperiodic homeomorphism. For minimal homeomorphisms the problem is solved as follows: every Cantor minimal system is homeomorphic to a Bratteli–Vershik model acting on a simple Bratteli diagram and vice versa. Every simple Bratteli diagram can be endowed with an ordering so that the Vershik map is well-defined and minimal [7]. The following proposition gives a particular solution of the above problem.
Proposition 2.5. Let $B = (V, E)$ be a Bratteli diagram such that every cofinal class is infinite and let $G$ be a locally finite group generating the cofinal equivalence relation on the path space of $B$. Suppose $G$ has only one minimal component. Then there exists an ordering on $B$ that admits the well-defined aperiodic Vershik map.

Notice that if the group $G$ in Proposition 2.5 has two minimal components, then a continuous Vershik map may not exist. For example, any ordering on the following diagram does not define a continuous Vershik map.

3. Applications

Having represented a Cantor aperiodic system as the Vershik map on an ordered Bratteli diagram, we can apply the technique of Bratteli diagrams developed in [4–7] to explore properties of the system. In this section, we generalize some results of [4,5] to aperiodic homeomorphisms with a finite number of minimal components.

Given a Cantor aperiodic system $(X, T)$, denote by $C_T$ the set of minimal $T$-components and set $\mathcal{N}(T) = \min\{\text{card}(Y) : Y$ is a basic set}.

Proposition 3.1. (i) $\mathcal{N}(T)$ is an invariant of orbit equivalence. (ii) $\text{card}(C_T) \leq \mathcal{N}(T)$. (iii) If $\text{card}(C_T) < \infty$, then $\text{card}(C_T) = \mathcal{N}(T)$.

The following proposition shows that Bratteli diagrams associated to aperiodic homeomorphisms with a finite number of minimal components can be chosen to be ‘almost’ simple.

Proposition 3.2. Let $(X, T)$ be a Cantor aperiodic system such that $\mathcal{N}(T) < \infty$. Then $(X, T)$ can be represented as the Vershik map acting on the path space of an ordered diagram $B$ so that $B$ has exactly $\mathcal{N}(T)$ simple invariant (with respect to the cofinal equivalence relation) subdiagrams and each of these subdiagrams has a unique maximal and minimal path.

Example 1. Consider the substitutional system $(X_\sigma, T_\sigma)$ given by $\sigma(a) = abab$, $\sigma(b) = abb$, and $\sigma(c) = accb$. Notice that $(X_\sigma, T_\sigma)$ is a non-minimal system with $\mathcal{N}(T) = 1$. Using the ideas from [3], one can prove that $\mathcal{X}_n = \{T_\sigma^i \sigma^j([w]) : w \in \{a, b, c\}, 0 \leq i < |\sigma^n([w])|, n \geq 1\}$, are K–R partitions satisfying the conditions of Theorem 2.1. Thus, $(X_\sigma, T_\sigma)$ is isomorphic to a Vershik map of the stationary diagram.

Clearly, this Bratteli diagram is non-simple and has only one simple invariant subdiagram consisting of paths that never go through $c$. See also Proposition 2.5.

Proposition 3.2 allows us to generalize some results proved earlier for Cantor minimal systems to the case of aperiodic homeomorphisms with a finite number of minimal components.

The following proposition generalizes Theorem 4.18 of [5]. Recall that an equivalence relation on a Cantor set is said to be affable if it is homeomorphic to the cofinal equivalence relation on a Bratteli diagram (for the details see [5]).

Proposition 3.3. If $(X, T)$ is a Cantor aperiodic system such that $\mathcal{N}(T) < \infty$, then the equivalence relation $R_T$ on $X$ generated by the orbits of $T$ is affable.
Given two equivalence relations $E$ and $F$ on Cantor sets $X$ and $Y$ respectively, we say that $E$ is embeddable into $F$, in symbols $E \subseteq F$, if there is a continuous injection $f : X \to Y$ such that $x Ey$ iff $f(x)Ff(y)$. And $E$ and $F$ are bi-embeddable if $E \subseteq F$ and $F \subseteq E$. An equivalence relation is called aperiodic if each equivalence class is infinite. The notion of bi-embeddability was originally considered in [2] for countable Borel equivalence relations.

**Theorem 3.4.** Any two aperiodic affable equivalence relations are bi-embeddable.

**The idea of the proof.** If we have two aperiodic Bratteli diagrams $B_1$ and $B_2$, i.e. every cofinal class is infinite, then by appropriate telescoping and microscoping (for the definitions see [4]) of $B_1$, we can see $B_2$ as a subdiagram of $B_1$ and vice versa. □

Consider a Cantor aperiodic system $(X, T)$ with a basic set $Y$. Let $C(X, \mathbb{Z})$ be the group of continuous functions from $X$ to $\mathbb{Z}$. Set $C(X|Y, \mathbb{Z}) = \{ f \in C(X, \mathbb{Z}) : f|_Y \equiv \text{const} \}$ and $\partial_T f = f \circ T - f$ for $f \in C(X, \mathbb{Z})$. Denote by $K^0(X|Y, T)$ the group $C(X, \mathbb{Z})/\partial_T C(X, \mathbb{Z})$, which is called a relative dimension group of $(X, T, Y)$ [6]. Denote by $K^0(X|Y, T)^+$ and $u_0$ the image of the positive cone $\{ f \in C(X, \mathbb{Z}) : f \geq 0 \}$ and $1$ in the quotient group $K^0(X|Y, T)$, respectively.

**Remark 1.** If a basic set $Y$ is a singleton, then $K^0(X|Y, T) = C(X, \mathbb{Z})/\partial_T C(X, \mathbb{Z})$ and we denote this group by $K^0(X, T)$.

The proof of the following proposition is based on Proposition 5.2 of [6].

**Proposition 3.5.** Let $(X, T)$ be a Cantor aperiodic system with a basic set $Y$.

(i) The relative dimension group $(K^0(X|Y, T), K^0(X|Y, T)^+)$ is an ordered group and $u_0$ is an order unit.

(ii) For every Bratteli diagram $B = (V, E)$ associated to the basic set $Y$, the dimension group $K_0(V, E)$ of the diagram $B$ (for the definition see [4,7]) is isomorphic (as an ordered dimension group with order unit) to $K^0(X|Y, T)$.

The correspondence between the dimension group of a Cantor dynamical system and the dimension group of the associated Bratteli diagram allows us to generalize Theorem 2.1 of [4].

**Corollary 3.6.** Let $(X, T)$ and $(Y, S)$ be Cantor aperiodic systems such that $N(T) = N(S) = 1$. Then $T$ and $S$ are strong orbit equivalent if and only if $K^0(X, T)$ and $K^0(Y, S)$ are isomorphic (as ordered dimension groups with order units).

**The sketch of the proof.** We notice only that if $K^0(X, T) \cong K^0(Y, S)$, then the unordered Bratteli diagrams constructed for homeomorphisms $T$ and $S$ by single-point basic sets are equivalent. To finish the proof we apply the arguments of Theorem 1.1 of [6] or Theorem 2.1 of [4]. □

**References**


