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Optimal Control

On the controllability for Maxwell's equations in specific media

Patrick Courilleau^a, Thierry Horsin Molinaro^b

^a Université de Cergy-Pontoise, département de mathématiques (UMR 8088), 2, avenue Adophe-Chauvin, 95302 Cergy-Pontoise cedex, France ^b Université de Versailles-St Quentin, laboratoire de mathématiques appliquées, 45, avenue des États-Unis, 78035 Versailles, France

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Abstract

In this Note we study the problem of exact controllability of the Maxwell's equations in specific media with two different models, on the one hand the so-called Drude–Born–Fedorov model, in the time domain, and on the other hand a simplified bilinear medium.

For the first one we prove the non approximate controllability whereas for the second one we are able to prove the exact controllability under the usual conditions of the wave equation. To cite this article: P. Courilleau, T. Horsin, C. R. Acad. Sci. Paris, Ser. I 341 (2005).

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Résumé

Sur la controlabilté des équations de Maxwell dans des milieux spécifiques. Dans cette Note nous étudions le problème de la contrôlabilité exacte des équations de Maxwell en milieux spécifiques avec deux différents modèles, d'un côté le modèle en temps de Drude–Born–Fedorov, et d'un autre côté un milieu bilinéaire simplifié.

Pour le premier on prouve la non contrôlabilité approchée alors que pour le second nous prouvons la contrôlabilité exacte sous les conditions usuelles suffisantes de l'équation des ondes. *Pour citer cet article : P. Courilleau, T. Horsin, C. R. Acad. Sci. Paris, Ser. I 341 (2005).*

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1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected open set with a connected boundary Σ and let T be a positive real number. We deal here with the following problem: find a (postponed defined) function $J:(0, T) \times \Sigma_0 \to \mathbb{R}^3$ where $\Sigma_0 \subset \Sigma$ such that the solution $\binom{E}{H}$ of the following system

$$\partial_t \left(A \begin{pmatrix} E \\ H \end{pmatrix} \right) + \begin{pmatrix} 0 & -\operatorname{curl} \\ \operatorname{curl} & 0 \end{pmatrix} \begin{pmatrix} E \\ H \end{pmatrix} = 0 \quad \text{in } \mathcal{D}(\Omega)^6,$$

$$H \wedge n = J \quad \text{on } (0, T) \times \Sigma_0,$$

$$B(E) = 0 \quad \text{on } (0, T) \times \Sigma$$
(1)

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E-mail addresses: patrick.courilleau@math.u-cergy.fr (P. Courilleau), horsin@math.uvsq.fr, thierry.horsin@math.uvsq.fr (T. Horsin).

can be driven from any initial data, $\binom{E_0}{H_0}$, at time t = 0 to the zero final data at time t = T. Here B(E) denotes a boundary condition on E and A denotes various operators that we will specify below. In the case of classical Maxwell's equations one has $A = \begin{pmatrix} \varepsilon & 0 \\ 0 & \mu \end{pmatrix}$ where ε , μ are positive constant real numbers, and we refer to [1,2,4,8,9] for a range of other models. Eventually Σ_0 is a measurable (possibly strict) subset of Σ .

Let us remark that for the use of convenient functional spaces for E and H, we will look for $J \in L^2(0, T, T)$ (H^{-1/2}_{div}(Σ₀)) (see for example [10] for the definition of this space).
• We consider first the case of the so-called DBF time approximation for which one has

$$A\begin{pmatrix} E\\ H \end{pmatrix} = \begin{bmatrix} \varepsilon(E+\beta\operatorname{curl} E)\\ \mu(H+\beta\operatorname{curl} H) \end{bmatrix}$$

In this model one takes usually $E \wedge n = 0$ on the boundary in the case when one looks for J satisfying, at least formally, div_{Σ} J = 0. Here β is a constant positive real number called the chiral number of the media.

We prove

Theorem 1.1. The DBF system is not approximately controllable, for values of β out of an at most countable set, in $H(\operatorname{div} = 0) \times H(\operatorname{div} = 0)$ with $E \wedge n = 0$ on Σ .

Here $H(\text{div} = 0) := \{u \in L^2(\Omega)^3, \text{div}(u) = 0 \text{ in } \mathcal{D}'(\Omega)\}$. We also denote $H(\text{curl}) := \{u \in L^2(\Omega)^3, \text{curl}(u) \in U^2(\Omega)\}$ $L^{2}(\Omega)^{3}$. Furthermore we denote $H_{0}(\text{div} = 0) = \{\varphi \in H(\text{div} = 0), \varphi \cdot n = 0\}$ and $H_{J}(\text{curl}) = \{\varphi \in H(\text{curl}), \varphi \in H(\text{curl})\}$ $\varphi \wedge n = J$.

In a work in preparation [6], we prove that in fact this result can be improved by considering time-harmonic solutions of this problem and then consider the controllability only in the space domain, for which we can prove the approximate controllability either in H or in E provided one add for the latter the condition $E \cdot n = 0$ on the boundary.

• Another typical model of medium is the so-called bilinear case where A takes the form $A\binom{E}{H} = \binom{\varepsilon E + \delta H}{\delta E + \mu H}$, and again the coefficients are assumed to be constant positive real numbers and satisfy a coercivity condition $\varepsilon \mu - \delta^2 > 0$. Here we have a simplified bilinear medium, since the general model with constant coefficients involves ε and μ to be linear operators. For the model considered here, we prove:

Theorem 1.2. The bilinear medium case is exactly controllable in $H(div = 0) \times H(div = 0)$ provided that the sufficient conditions for the controllability of the wave equation are satisfied by T and Σ_0 .

We generalize this result in a work in preparation [5] for bilinear delayed media.

2. Proof of Theorem 1.1

It is proven in [3] that the system (1) is well posed provided one takes E_0 and H_0 in $H_0^1(\Omega)$ and J in $H^{1/2}(\Sigma_0)$ and the solution is sought in

 $C([0, T]; H(\text{div} = 0) \cap H_0(\text{curl})) \times C([0, T]; H(\text{div} = 0) \cap H_J(\text{curl})).$

So here we take $B(E) = E \wedge n$.

To prove Theorem 1.1 we need to prove that the adjoint system does not satisfy the unique continuation property. Let φ_0 and ψ_0 be in H(div = 0) and consider the formal adjoint system

$$\begin{cases} \varepsilon(\psi_t + \beta \operatorname{curl} \psi_t) - \operatorname{curl} \varphi = 0, \\ \mu(\varphi_t + \beta \operatorname{curl} \varphi_t) + \operatorname{curl} \psi = 0, \\ (\psi + \mu \beta \varphi_t) \cdot n = 0 \quad \text{on } \Sigma_0 \times [0, T], \\ \varphi_{t=0} = \varphi_0, \quad \psi_{t=0} = \psi_0. \end{cases}$$
(2)

Then one easily checks that with these notations one has formally

$$-\int_{0}^{I}\int_{\Sigma_{0}} J \cdot (\psi + \mu\beta\varphi_{t}) = \int_{\Omega} \varepsilon(E_{0} + \beta\operatorname{curl} E_{0})\psi_{0} \,\mathrm{d}x + \int_{\Omega} \mu(H_{0} + \beta\operatorname{curl} H_{0})\varphi_{0} \,\mathrm{d}x.$$
(3)

Let φ_0 and ψ_0 two elements of $H_0^1(\Omega)$. We admit for a while that there exist unique solutions φ and ψ in $C([0, T]; H_0^1(\Omega))$ of

$$\begin{cases} \varepsilon(\psi_t + \beta \operatorname{curl} \psi_t) - \operatorname{curl}(\varphi) = 0, \\ \mu(\varphi_t + \beta \operatorname{curl} \varphi_t) + \operatorname{curl}(\psi) = 0, \\ \varphi_{t=0} = \varphi_0, \quad \psi_{t=0} = \psi_0. \end{cases}$$
(4)

The tangential part of $(\psi + \mu\beta\varphi_t)$ is equal to 0 on Σ_0 with non trivial functions, thus the last quantity is not a norm on the space of initial data which is the necessary and sufficient condition for the controllability. Even more, this proves that we don't have approximate controllability.

To prove the existence and uniqueness of solutions of (4) one can use [3] as well as the following argument. The continuous operator $I + \beta$ curl is invertible from $H_0(\text{curl}) \cap H^1$ into H(div = 0) for values of β in the complement of a countable set (see also [7]). One can easily check that if $u \in H_0(\text{curl})$ then $\text{curl}(u) \in H_0(\text{div} = 0)$. Thus $I + \beta$ curl define a invertible bounded operator between H_0^1 and $H_0(\text{div} = 0)$. The system given in (4) is equivalent to a system of Cauchy–Lipschitz type for functions f and g in $H_0(\text{div} = 0)$ given by

$$\begin{cases} \varepsilon f_t - \operatorname{curl}((I + \beta \operatorname{curl})^{-1}(g)) = 0, \\ \mu g_t + \operatorname{curl}((I + \beta \operatorname{curl})^{-1}(f)) = 0 \end{cases}$$
(5)

and one has the existence and uniqueness of solutions.

3. Proof of Theorem 1.2

In order to prove this theorem, for any φ_0 and ψ_0 in H(div = 0), let φ and ψ satisfy

$$\mu\varphi_t + \delta\psi_t - \operatorname{curl}\psi = 0,\tag{6}$$

$$\varepsilon\psi_t + \delta\varphi_t + \operatorname{curl}\varphi = 0,\tag{7}$$

$$\varphi \wedge n = 0 \quad \text{on } \Sigma,$$
(8)

$$(\varepsilon\psi + \delta\varphi) \cdot n = 0 \quad \text{on } \Sigma_0, \tag{9}$$

$$\varphi_{t=0} = \varphi_0, \quad \psi_{t=0} = \psi_0. \tag{10}$$

One has the following formal identity for any solutions of (1) vanishing at time t = T:

$$\int_{0}^{1} \int_{\Sigma_{0}} J \cdot \psi \, \mathrm{d}\sigma \, \mathrm{d}t = \int_{\Omega} \left(\psi_{0}(\varepsilon E_{0} + \delta H_{0}) + \varphi_{0}(\delta E_{0} + \mu H_{0}) \right) \mathrm{d}x.$$

Taking $J = \varepsilon \psi + \delta \varphi$ in (1) that we solved backward on (0, T) with $E_{t=T} = H_{t=T} = 0$, we can define Λ that maps (ψ_0, φ_0) to $(E_0, H_0) = (E_{t=0}, H_{t=0})$ which satisfies

$$\int_{0}^{1} \int_{\Omega} \varepsilon \left| \psi + \frac{\delta}{\varepsilon} \varphi \right|^{2} d\sigma \, dt = \left\langle A \begin{pmatrix} \psi_{0} \\ \varphi_{0} \end{pmatrix}, A \begin{pmatrix} \psi_{0} \\ \varphi_{0} \end{pmatrix} \right\rangle. \tag{11}$$

Note that we have used (8).

One checks that $(\varphi, \varepsilon \psi + \delta \varphi)$ satisfies the system

$$\begin{cases} (\varepsilon \mu - \delta^2) u_t - \operatorname{curl} v = 0, \\ v_t + \operatorname{curl} u = 0, \\ u \wedge n = 0 \quad \text{on } \Sigma, \\ v \cdot n = 0 \quad \text{on } \Sigma_0. \end{cases}$$
(12)

Now referring to [11] and since $(\varepsilon \mu - \delta^2) > 0$, one knows that the left-hand side of (11) defines a norm on the space of initial data provided that *T* is large enough and Σ_0 satisfies some geometric properties and in the same time it proves the existence and uniqueness of solutions of (6)–(10).

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