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Mathematical Problems in Mechanics

Equivalence estimates for a class of singular perturbation problems ☆

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Abstract

We give some equivalence estimates on the solution of a singular perturbation problem that represents, among other models, the Koiter and Naghdi shell models. Two of the estimates apply to intermediate shell problems and the third is for membrane/shear dominated shells. From these equivalences, many known and some new sharp estimates on the solutions of the singular perturbation problems easily follow. ***To cite this article: S. Zhang, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Estimations d'équivalence pour une classe de problèmes de perturbations singulières. Nous donnons des estimations d'équivalence de la solution d'un problème de perturbations singulières pour des modèles de coques qui englobent les modèles de Koiter et de Naghdi. Deux de ces estimations sont valables pour les problèmes de coques dits intermédiaires, la troisième s'applique à des coques de type membrane/cisaillement. Quelques unes de ces équivalences sont connues, mais d'autres équivalences donnent des résultats précis pour des solutions de problèmes de perturbations singulières. ***Pour citer cet article : S. Zhang, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

Let H, U, and V be Hilbert spaces, A and B be linear continuous operators from H to U and V, respectively. We consider the problem: Given $f \in H^*$, the dual space of H, and $\epsilon > 0$, find $u^\epsilon \in H$, such that

$$\epsilon^2 (Au^\epsilon, Av)_U + (Bu^\epsilon, Bv)_V = \langle f, v \rangle \quad \forall v \in H. \tag{1}$$

We will use the notations $P \simeq Q$ and $P \lesssim Q$, which mean that there exist constants C_1, C_2, and C independent of ϵ, P, and Q such that $C_1 P \leqslant Q \leqslant C_2 P$ and $P \leqslant CQ$, respectively. We assume

$$\|Au\|_U + \|Bu\|_V \simeq \|u\|_H \quad \forall u \in H \tag{2}$$

such that (1) has a unique solution $u^\epsilon \in H$. Further, we assume that $\ker B = \{0\}$ and the range of B, denoted by $W = B(H)$, is dense in V but not equal to V, so (1) is a singular perturbation problem. This problem represents

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the Koiter and Naghdi models of shells that inhibit pure bending deformations, for which the equivalence (2) can be found in [7]. The function $\|B\cdot\|_V$ defines a weaker norm on H. If the functional f is continuous with respect to this norm, the shell problem is membrane/shear dominated. Otherwise, it is intermediate. In this note, we establish some equivalent estimates on u^ϵ for both of the two cases. The proofs of these equivalences are very simple, see Section 2. From these estimates, a number of old and new sharp estimates on the behavior of u^ϵ easily follow. Among other things, we show in Section 3 that u^ϵ converges to a limit in a norm at a rate in any case. (The norm is problem dependent, and it could be very weak.) Such convergence occurs at a higher rate in the membrane/shear dominated case.

2. Equivalent estimates

We recall some terminologies in Hilbert spaces. For a Hilbert space X, we denote its dual by X^*, as we already did, and for any $f\in X^*$, we use $i_X f\in X$ to denote its Riesz representation. The isomorphism $\pi_X : X\to X^*$ is defined as the inverse of i_X. If X and Y are Hilbert spaces with $X\subset Y$, and if X is dense in Y, then the restriction operator defines an injection of Y^* onto a dense subspace of X^* (and we identify Y^* with that dense subspace). For such pair of Hilbert spaces, we have $X\cap Y=X$ and $X+Y=Y$ as set, on which we define new norms by $\|z\|_{X\cap Y}=(\|z\|_X^2+\|z\|_Y^2)^{1/2}$ and $\|z\|_{X+Y}=\inf_{z=x+y}(\|x\|_X^2+\|y\|_Y^2)^{1/2}$, respectively. With these norms, the intersection $X\cap Y$ and the sum $X+Y$ are themselves Hilbert spaces. The dual spaces X^* and Y^* can be viewed as subspace of $(X\cap Y)^*$ and we have [3] $X^*+Y^*=(X\cap Y)^*$. Associated with a Hilbert space X and any positive number ϵ, we define the Hilbert space ϵX. As set, ϵX equals to X, but the norm is defined by $\|x\|_{\epsilon X}=\epsilon\|x\|_X$. We have $(\epsilon X)^*=\epsilon^{-1}X^*$, $(\epsilon X\cap Y)^*=\epsilon^{-1}X^*+Y^*$, and $(\epsilon X+Y)^*=\epsilon^{-1}X^*\cap Y^*$. The K-functional [3] on such Hilbert couple $[Y,X]$ is a norm on $X+Y$ defined by $K(\epsilon,y,[Y,X])=\inf_{y=y_1+y_2,\ y_1\in Y,\ y_2\in X}\{\|y_1\|_Y+\epsilon\|y_2\|_X\}$. Obviously, $K(\epsilon,y,[Y,X])\simeq\|y\|_{Y+\epsilon X}$. Since X is dense in Y, we have $\lim_{\epsilon\to 0}K(\epsilon,y,[Y,X])=0\ \forall y\in Y$. Furthermore, if $y\in[Y,X]_{\theta,p}$ (the real interpolation space based on the K-functional) for some $1\leqslant p\leqslant\infty$ and $0<\theta\leqslant 1$, then $K(\epsilon,y,[Y,X])\lesssim\epsilon^\theta\|y\|_{[Y,X]_{\theta,p}}$.

We define a norm on W by the function $\|B^{-1}\cdot\|_H$, then the operator B is an isomorphism between H and W. For any $f\in H^*$, there is a unique $\xi^*\in W^*$ such that $\langle f,v\rangle=\langle\xi^*,Bv\rangle\ \forall v\in H$. Let $\overline{H}$ be the completion of H with respect to the $\|B\cdot\|_V$ norm. Then B can be extended to $\overline{H}$ to define an isomorphism, denoted by $\overline{B}$, between $\overline{H}$ and V. The function $\|\pi_V B\cdot\|_{W^*}$ also defines a norm on H which we call the $\overline{\overline{H}}$ norm. It is weaker than the $\overline{H}$ norm. We denote the completion of H with respect to this new norm by $\overline{\overline{H}}$. The operator $\pi_V B : H\to W^*$ can then be uniquely extended to $\overline{\overline{H}}$, and the extension, denoted by $\overline{\pi_V B}$, is an isomorphism between $\overline{\overline{H}}$ and W^*. Thus for $\xi^*\in W^*$, there is a unique $u^0\in\overline{\overline{H}}$ such that $\overline{\pi_V B}u^0=\xi^*$. We will see that this u^0 is the limit of u^ϵ in $\overline{\overline{H}}$. If $f\in\overline{H}^*$, i.e., f is continuous with respect to the $\|B\cdot\|_V$ norm, then $\xi^*\in V^*$ and $u^0\in\overline{H}$ satisfies $\overline{B}u^0=\xi=i_V\xi^*$. We have $K(\epsilon,f,[H^*,\overline{H}^*])=K(\epsilon,u^0,[\overline{\overline{H}},\overline{H}])=K(\epsilon,\xi^*,[W^*,V^*])\simeq\|\xi^*\|_{W^*+\epsilon V^*}$. And if $f\in\overline{H}^*$, $K(\epsilon,u^0,[\overline{H},H])=K(\epsilon,\xi,[V,W])\simeq\|\xi\|_{V+\epsilon W}$. The following two theorems are the main results of this note.

Theorem 2.1. *For the solution of the problem* (1), *we have the equivalences*

$$\epsilon\|Au^\epsilon\|_U+\|Bu^\epsilon\|_V\simeq\epsilon^{-1}\|\xi^*\|_{W^*+\epsilon V^*} \tag{3}$$

and

$$\|\pi_V Bu^\epsilon-\xi^*\|_{W^*}+\epsilon\|Bu^\epsilon\|_V\simeq\|\xi^*\|_{W^*+\epsilon V^*}. \tag{4}$$

Proof. Since

$$\epsilon^2(Au^\epsilon,Av)_U+(Bu^\epsilon,Bv)_V=\langle\xi^*,Bv\rangle\quad\forall v\in H, \tag{5}$$

we have

$$\begin{aligned}\epsilon^2(Au^\epsilon,Au^\epsilon)_U+(Bu^\epsilon,Bu^\epsilon)_V&=\langle\xi^*,Bu^\epsilon\rangle\lesssim\|\xi^*\|_{\epsilon^{-1}W^*+V^*}\|Bu^\epsilon\|_{\epsilon W\cap V}\\&\simeq\|\xi^*\|_{\epsilon^{-1}W^*+V^*}\big(\epsilon\|Bu^\epsilon\|_W+\|Bu^\epsilon\|_V\big)\lesssim\|\xi^*\|_{\epsilon^{-1}W^*+V^*}\big(\epsilon\|Au^\epsilon\|_U+\|Bu^\epsilon\|_V\big).\end{aligned}$$

Thus $\epsilon\|Au^\epsilon\|_U+\|Bu^\epsilon\|_V\lesssim\epsilon^{-1}\|\xi^*\|_{W^*+\epsilon V^*}$. On the other hand,

$$\epsilon^{-1}\|\xi^*\|_{W^*+\epsilon V^*}=\sup_{v\in H}\frac{\langle\xi^*,Bv\rangle}{\|Bv\|_{\epsilon W\cap V}}=\sup_{v\in H}\frac{\epsilon^2(Au^\epsilon,Av)_U+(Bu^\epsilon,Bv)_V}{\|Bv\|_{\epsilon W\cap V}}\lesssim\epsilon\|Au^\epsilon\|_U+\|Bu^\epsilon\|_V.$$

The equivalence (3) then follows. We write Eq. (5) as $\langle \pi_V Bu^\epsilon - \xi^*, Bv\rangle = -\epsilon^2(Au^\epsilon, Av)_U\ \forall v \in H$. From this we see

$$\|\pi_V Bu^\epsilon - \xi^*\|_{W^*} = \sup_{v\in H} \frac{|\langle \pi_V Bu^\epsilon - \xi^*, Bv\rangle|}{\|Bv\|_W} = \sup_{v\in H} \epsilon^2 \frac{|(Au^\epsilon, Av)_U|}{\|Bv\|_W} \lesssim \epsilon^2 \|Au^\epsilon\|_U. \tag{6}$$

Therefore $\|\pi_V Bu^\epsilon - \xi^*\|_{W^*} + \epsilon \|Bu^\epsilon\|_V \lesssim \epsilon^2 \|Au^\epsilon\|_U + \epsilon \|Bu^\epsilon\|_V$. One direction of (4) then follows from this estimate and (3). The other direction of (4) follows from the definition of the sum norm. □

Theorem 2.2. *If $f \in \overline{H}^*$, i.e., $\xi^* \in V^*$, then for the solution u^ϵ of* (1), *the equivalence*

$$\epsilon \|u^\epsilon\|_H + \|Bu^\epsilon - \xi\|_V + \epsilon^{-1} \|\pi_V Bu^\epsilon - \xi^*\|_{W^*} \simeq \|\xi\|_{\epsilon W+V} \tag{7}$$

holds. Here $\xi = i_V \xi^*$ is the Riesz representation of ξ^*.

Proof. When $\xi^* \in V^*$, we have $\epsilon^2(Au^\epsilon, Au^\epsilon)_U + (Bu^\epsilon, Bu^\epsilon)_V = (\xi, Bu^\epsilon)_V$. Thus

$$\epsilon^2(Au^\epsilon, Au^\epsilon)_U + (Bu^\epsilon - \xi, Bu^\epsilon - \xi)_V = -(Bu^\epsilon - \xi, \xi)_V = -\langle \pi_V Bu^\epsilon - \xi^*, \xi\rangle.$$

The right-hand side dual product can be bounded as follows.

$$\begin{aligned}\big|\langle \pi_V Bu^\epsilon - \xi^*, \xi\rangle\big| &\leqslant \|\pi_V Bu^\epsilon - \xi^*\|_{\epsilon^{-1} W^* \cap V^*} \|\xi\|_{\epsilon W+V} \\ &\simeq \epsilon^{-1} \|\pi_V Bu^\epsilon - \xi^*\|_{W^*} \|\xi\|_{\epsilon W+V} + \|Bu^\epsilon - \xi\|_V \|\xi\|_{\epsilon W+V} \\ &\lesssim \big(\epsilon \|Au^\epsilon\|_U + \|Bu^\epsilon - \xi\|_V\big) \|\xi\|_{\epsilon W+V}.\end{aligned}$$

In the last step we used (6). Thus $\epsilon \|Au^\epsilon\|_U + \|Bu^\epsilon - \xi\|_V \lesssim \|\xi\|_{\epsilon W+V}$. From this we see that $\|Bu^\epsilon\|_V \lesssim \|\xi\|_V + \|\xi\|_{\epsilon W+V} \lesssim \|\xi\|_V$. Therefore, $\epsilon \|u^\epsilon\|_H + \|Bu^\epsilon - \xi\|_V \lesssim \|\xi\|_{\epsilon W+V}$. On the other hand, from the definition of the sum norm, we see that $\|\xi\|_{\epsilon W+V} \leqslant \epsilon \|Bu^\epsilon\|_W + \|Bu^\epsilon - \xi\|_V \lesssim \epsilon \|u^\epsilon\|_H + \|Bu^\epsilon - \xi\|_V$. From (6) we see $\|\pi_V Bu^\epsilon - \xi^*\|_{W^*} \lesssim \epsilon^2 \|Au^\epsilon\|_U \lesssim \epsilon \|\xi\|_{\epsilon W+V}$. □

3. Applications

In view of the properties of the K-functional many useful estimates, convergences, and convergence rates directly follow from the estimates established in the last section. First, the equivalence (3) shows that the magnitude of the energy $E(\epsilon) := \epsilon^2(Au^\epsilon, Au^\epsilon)_U + (Bu^\epsilon, Bu^\epsilon)_V$ is totally determined by the K-functional of f. Namely, $E(\epsilon) \simeq \epsilon^{-2} K^2(\epsilon, f, [H^*, \overline{H}^*])$. This relationship seems new and it immediately yields many sharp estimates on the energy given in [1,2,4]. We also see the well-known results like $E(\epsilon) = \mathrm{o}(\epsilon^{-2})$ and $E(\epsilon)$ is bounded iff $f \in \overline{H}^*$, or the shell problem is membrane/shear dominated, see [11] for example.

From the equivalence (4) we see that

$$\big\|u^\epsilon - u^0\big\|_{\overline{\overline{H}}} + \epsilon \|u^\epsilon\|_{\overline{H}} \simeq K\big(\epsilon, f, \big[H^*, \overline{H}^*\big]\big). \tag{8}$$

It follows that u^ϵ converges to the limit u^0 in $\overline{\overline{H}}$. An equivalent result can be found in [5]. The equivalence (8) suggests that without additional assumption on f the topology of $\overline{\overline{H}}$ is the strongest in which u^ϵ converges. However, the topology of $\overline{\overline{H}}$ could be very weak, see [8,9], and [11]. When $f \in [H^*, \overline{H}^*]_{\theta,\infty}$ for some $\theta \in (0, 1]$ we have the convergence rate $\|u^\epsilon - u^0\|_{\overline{\overline{H}}} \lesssim \epsilon^\theta$. This convergence rate is solely determined by the 'classification-index [4]' of an intermediate shell. For example for the Scordelis–Lo roof [6] we have $\|u^\epsilon - u^0\|_{\overline{\overline{H}}} \lesssim \epsilon^{5/8}$. By interpolation of (8) and (9) below, convergence of u^ϵ in stronger norms can be obtained if f is more 'regular'.

From the equivalence (7) we see that when $f \in \overline{H}^*$

$$\epsilon \|u^\epsilon\|_H + \big\|u^\epsilon - u^0\big\|_{\overline{H}} + \epsilon^{-1}\big\|u^\epsilon - u^0\big\|_{\overline{\overline{H}}} \simeq K\big(\epsilon, u^0, [\overline{H}, H]\big) \simeq K\big(\epsilon, \xi, [V, W]\big). \tag{9}$$

Thus we have the convergence $\lim_{\epsilon\to 0} \|u^\epsilon - u^0\|_{\overline{H}} = 0$, as proved for membrane shells [7]. In this case, we have the faster convergence $\|u^\epsilon - u^0\|_{\overline{\overline{H}}} = \mathrm{o}(\epsilon)$, which seems a new result. Furthermore, if $\xi = i_V \xi^* \in [V, W]_{\theta,\infty}$ (i.e., $u^0 \in [\overline{H}, H]_{\theta,\infty}$) for some $\theta \in (0, 1]$ (for clamped elliptic shell, $\theta = 1/6$ [7]) we have the rate estimate $\epsilon \|u^\epsilon\|_H + \|u^\epsilon - u^0\|_{\overline{H}} + \epsilon^{-1} \|u^\epsilon - u^0\|_{\overline{\overline{H}}} \lesssim \epsilon^\theta$. The above results can be used to obtain sharp estimates on many elliptic-elliptic singular perturbation problems, see [10] and [12].

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