Algebraic Geometry

Twisted Chern classes and $\mathbb{G}_m$-gerbes

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Abstract

Using the language of stacks one can give a simple definition of functorial Chern classes for twisted sheaves. Calculating the cohomology ring of a $\mathbb{G}_m$-gerbe we observe that the twisted Chern classes used by Huybrechts and Stellari are specializations of these classes. We describe explicitly the relation between the choice of a cocycle in the definition of twisted sheaves and the 2-categorical structure of $\mathbb{G}_m$-gerbes.

1. Introduction

Recently, Huybrechts and Stellari defined cohomological Chern classes for twisted sheaves on smooth projective varieties and used these in the study of derived equivalences on K3 surfaces [5,6]. In these articles the authors also point out that the new Chern classes for twisted sheaves do not behave as functorially as one would expect. We note that these problems disappear if one uses the language of $\mathbb{G}_m$-gerbes. In particular, in this setup, Chern classes take values in the cohomology of a gerbe over a space $X$, which is a polynomial ring over $H^*(X, \mathbb{Q})$ if the class of the gerbe is torsion. Furthermore, the choice of a cocycle appearing in the definition of twisted sheaves in loc. cit. can be explained by the fact that gerbes form a 2-category, whose structure is given by a truncated cohomology complex instead of a cohomology group.

These results are probably well known to specialists, but since I could not find them in the literature I thought that it might be useful to give a short explanation of these facts.

In the following $X$ is either a variety over $\mathbb{C}$ or a differentiable manifold. In the differentiable setting the results also hold if one replaces the multiplicative group $\mathbb{G}_m$ by $S^1$. 

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2. Cohomological Chern classes for vector bundles on stacks

Let \( E \) be a vector bundle of rank \( r \) on \( X \). It defines a map \( f_E: X \to \text{BGL}_r = [pt/\text{GL}_r] \) to the classifying stack (or space) of \( \text{GL}_r \)-bundles. Since \( H^*(\text{BGL}_r, \mathbb{Z}) \cong \mathbb{Z}[c_1, \ldots, c_r] \) (cf. \cite{2}) one can define cohomological Chern classes of \( E \) as \( c_i(E) = f_E^*(c_i) \). By descent for vector bundles the same holds for any algebraic stack (in the sense of Artin) or any differentiable stack \( \mathcal{M} \), i.e. any vector bundle \( E \) on \( \mathcal{M} \) defines a map \( \mathcal{M} \to \text{BGL}_r \). To describe this map explicitly in terms of groupoids, one can choose an atlas \( \pi: X \to \mathcal{M} \) such that \( \pi^*E \) is trivial. After trivializing, the descent datum for \( E \) is a map \( X \times_{\mathcal{M}} X \to \text{GL}_r \) and thus defines a map of groupoids \( (X \times_{\mathcal{M}} X \cong X) \to (\text{GL}_r \tilde{\times} pt) \) which defines a map \( \mathcal{M} \to \text{BGL}_r \).

We can again define cohomological Chern classes \( c_i(E) := f_E^*(c_i) \in H^*(\mathcal{M}, \mathbb{Z}) \).

3. Cohomology of \( \mathbb{G}_m \)-gerbes

Given a class \( \tau \in H^2(X, \mathbb{G}_m) = H^2(X, \mathcal{O}_X^*) \) we choose a \( \mathbb{G}_m \)-gerbe \( X^\tau \to X \) over \( X \) in this class (the relation between this choice and the choice of a cocycle for \( \tau \) is explained in the last section). Let \( d(\tau) \in H^3(X, \mathbb{Z}) \) be the Dixmier–Douady class of \( \tau \), given by the boundary of the exponential sequence.

**Lemma 3.1.** Let \( \pi: X^\tau \to X \) be a \( \mathbb{G}_m \)-gerbe over a variety and \( E \) a vector bundle of weight 1 on \( X^\tau \). Then \( H^*(X^\tau, \mathbb{Q}) \cong H^*(X, \mathbb{Q})[z] \), where \( z \in H^2(X, \mathbb{Q}) \) is the first Chern class of \( E \).

The same holds in the differentiable setting if \( \tau \in H^2(X, S^1)_{\text{tors}} \) is a torsion element.

Recall that a vector bundle \( E \) on \( X^\tau \) is of weight 1, if the \( \mathbb{G}_m \)-automorphisms of all points act by scalar multiplication on the fibres of \( E \), equivalently \( E \) is a \( \tau \)-twisted vector bundle on \( X \).

**Proof.** The Leray spectral sequence for \( \pi: E_2^{p,q} = H^p(X, R^q\pi_*\mathbb{Q}) \Rightarrow H^{p+q}(X^\tau, \mathbb{Q}) \), degenerates, since \( \tau^c \in R\pi_*\mathbb{Q} \) is the same as the cohomology of a projective bundle: We define \( E' \) := \( E^\oplus(c+1) \), which is a vector bundle of weight 1 on \( X^\tau \), \( \text{rk}(E') > c \) and \( c_1(E') = (c+1)c_1(E) \). Denote by \( s_0: X^\tau \to E' \) the zero section of \( E' \). Then \( p: E' - s_0(X^\tau) \to X^\tau \) is a vector bundle of weight 1, \( \text{rk}(E') \leq 2 \), acyclic (i.e., \( R^i\pi_*\mathbb{Q} = 0 \) for \( 0 < i < 2 \text{rk}(E') - 1 \)), therefore \( H^*(E' - s_0(X^\tau), \mathbb{Q}) \cong H^*(X^\tau, \mathbb{Q}) \) for \( * < 2 \text{rk}(E') - 1 \). Since \( E' \) is of weight 1, \( \pi: E' - s_0(X^\tau) \to X \) is a bundle of projective spaces and moreover the class \( \pi^*(c_1(E')) \) gives a generator for the rational cohomology of the fibres of \( \pi \). Thus \( R^1\pi_*\mathbb{Q} \) are constant sheaves and the spectral sequence for the cohomology of the projective bundle degenerates by the theorem of Leray–Hirsch. □

**Remark 1.** The class \( z \) generating the cohomology of the gerbe as algebra over \( H^*(X, \mathbb{Q}) \) depends on the choice of the vector bundle \( E \) of weight 1 on \( X^\tau \). For example, if we tensor \( E \) with the pull back of a line bundle \( L \) on \( X \) we change \( z \) by adding \( \pi^*(c_1(L)) \).

3.1. Remark on integral coefficients

The lemma does not hold for integral coefficients. It also fails in the differentiable setting, if the Dixmier–Douady class \( d(\tau) \) of the gerbe is not torsion: The Leray spectral sequence still looks like:

\[
\begin{array}{ccccccc}
H^0(X, \mathbb{Z}) & \cdots & & & & & \\
0 & 0 & d_3 & 0 & 0 & \cdots \\
H^0(X, \mathbb{Z}) & H^1(X, \mathbb{Z}) & H^2(X, \mathbb{Z}) & H^3(X, \mathbb{Z}) & \cdots
\end{array}
\]

where \( d_3 \) is the first differential that can be non-trivial. Since the gerbe is neutral over itself we have \( \pi^*(d(\tau)) = 0 \), therefore \( d(\tau) \) must lie in the image of \( d_3 \). Thus \( H^0(X, \mathbb{Z}) \not\cong \mathbb{Z} \) so that the locally constant sheaf \( R^2\pi_*\mathbb{Z} \) must be constant.

Since the spectral sequence is multiplicative, this shows that for twists such that \( d(\tau) \) is not a torsion class, the cohomology of \( X^\tau \) only contains a quotient of \( H^*(X, \mathbb{Z}) \).

More precisely, the differential \( d_3 \) maps a generator of \( H^0(X, \mathbb{Z}) \cong \mathbb{Z} \) to \( d(\tau) \). This can be seen by looking at the exponential sequence:
4. Comparison with the twisted Chern character of Huybrechts and Stellari

In [5], Huybrechts and Stellari define a twisted Chern character for a twist $\tau \in H^2(X, \mathbb{G}_m)$ depending on the choice of a class $B \in H^2(X, \mathbb{Q})$ with $\exp(B^{0,2}) = \tau$. Twisted bundles on $X$ are the same as bundles of weight 1 on the corresponding gerbe and their construction gives a specialization of the Chern classes defined above as follows.

First note that the exponential sequence: $H^2(X, \mathcal{O}_X) \xrightarrow{\exp} H^2(X, \mathcal{O}_X^*) \xrightarrow{d} H^3(X, \mathbb{Z})$ shows that if $\tau = \exp(B^{0,2})$ then the Dixmier–Douady class $d(\tau) = 0$. In particular, the twist $\tau$ is trivial in the differentiable category (the sheaf of differentiable functions is acyclic).

Thus there exists a differentiable line bundle $L$ of weight 1 on $X^\tau$ (equivalently a twisted line bundle $L$ on $X$). Then for any vector bundle $E$ of weight 1 on $X^\tau$ the tensor product $E \otimes L^{-1}$ is of weight 0. Therefore $E \otimes L^{-1}$ descends to a bundle on $X$ where one can apply the usual Chern character. Call the resulting class $\text{ch}_L(E)$.

Alternatively, the choice of $L$ defines a class $z = z(L) \in H^2(X^\tau, \mathbb{Q})$ and $\text{ch}_L(E)$ is obtained by setting $z = 0$ in the Chern character of $E$ on $X^\tau$ considered as a power series in $z$ ($\text{ch}(E) = \text{ch}(E \otimes L^{-1}) \exp(z)$ and the first factor lies in $H^*(X, \mathbb{Q}) \subset H^*(X^\tau, \mathbb{Q})$).

Huybrechts and Stellari point out that a canonical choice for the Chern class of $L$ is already determined by $B$: Indeed, denote by $S^1_{\text{diff}}$ (resp. $R_{\text{diff}}$) the sheaf of differentiable sections with values in $S^1$ (resp. in $R$). To define $X^\tau$ by $\tau$ we have to choose a 3-cocycle $\tau_{ijk} \in Z^3(X, \mathcal{O}_X^*)$ and since $\tau = \exp(B^{0,2})$ we can even choose $\tau_{ijk} \in Z^3(X, S^1_{\text{diff}})$. This can be done through a choice of a cocycle for $B$ in $Z^3(X, R)$. We have:

$$
\begin{align*}
Z^3(X, R) &\longrightarrow H^2(X, R) \\
\downarrow & \\
Z^2(X, R_{\text{diff}}) \xrightarrow{\exp} C^2(X, R_{\text{diff}}) \xrightarrow{d} Z^3(X, R_{\text{diff}}) \longrightarrow 0 \\
\downarrow & \\
Z^2(X, S^1_{\text{diff}}) \xrightarrow{\exp} C^2(X, S^1_{\text{diff}}) \xrightarrow{d} Z^3(X, S^1_{\text{diff}}) &\longrightarrow H^2(X, S^1_{\text{diff}}).
\end{align*}
$$

Since $R_{\text{diff}}$ is acyclic, the cocycle for $B$ is in fact a boundary in $Z^3(X, R_{\text{diff}})$. The choice of a lifting of $B$ to an element $a \in C^2(X, R_{\text{diff}})$ defines (by exp) a bundle $L$ of weight 1 on $X^\tau$. Two such liftings differ by an element of $Z^2(X, R_{\text{diff}}) = d(C^1(X, R_{\text{diff}}))$. Thus the Chern class of the bundle $L$ does not depend on the choice of $a$ and $B_{ijk}$.

Huybrechts and Stellari define the Chern class by $\text{ch}_B(E) := \text{ch}_L(E)$.

**Remark 2.** The Chern character defined on the gerbe $X^\tau$ has the advantage that it is compatible with morphisms of gerbes and does not depend on additional choices. Our approach compares to the one of Huybrechts and Stellari as follows: $\tau \in H^2(X, \mathbb{G}_m)$ corresponds to an isomorphism class of gerbes. The choice of the cocycle incorporates the choice of a gerbe in this isomorphism class (see Section 5). Note that in the applications the gerbe is usually canonical, typically defined by some moduli problem. Finally to get Chern classes with values in $H^*(X, \mathbb{Q})$, the choice of a $B$-field corresponds to the choice of a vector bundle of weight 1.

5. Choosing cocycles and the 2-category of gerbes

It is well-known that the set of isomorphism classes of $\mathbb{G}_m$-gerbes on $X$ is naturally isomorphic to $H^2(X, \mathbb{G}_m)$ (e.g., [3,1]). Stacks form a 2-category, and there is no canonical choice of such an isomorphism class. Thus, to define twisted sheaves on $X$ one usually chooses a cocycle for a given cohomology class $\tau \in H^2(X, \mathbb{G}_m)$. This choice also
determines a gerbe and the relation to the 2-categorical structure can be explained by recalling the proof of the result quoted above (the arguments work for any Abelian group instead of $\mathbb{G}_m$).

Let $0 \to A_0 \to A_1 \to A_2 \to 0$ be a complex of Abelian groups representing the complex $\tau_{\leq 2}R\Gamma(X, \mathbb{G}_m)$.

In the same way as a two term complex $A_0 \to A_1$ defines a groupoid (just because $A_0$ acts on $A_1$) one can define a 2-category from the above 3-term complex: The objects are elements in $A_2$; an object in $\text{Hom}(x, y)$ is an element $\phi \in A_1$ with $x + d(\phi) = y$ and a morphism $\Phi: \phi \to \psi$ is an element $\Phi \in A_0$ with $\phi + d(\Phi) = \psi$.

**Proposition 5.1.** The 2-category of $\mathbb{G}_m$-gerbes is equivalent to the two category defined by any 3-term complex representing $\tau_{\leq 2}R\Gamma(X, \mathbb{G}_m)$.

For 2-categories and equivalences between them see [4], Section 1. The proof below is the standard proof [1], keeping track of the morphisms.

**Proof.** First, we use a particular complex $A_\bullet$ as above and construct:

- For any element $a_2$ in $A_2$ a gerbe $X^{a_2} \to X$.
- For any $a_2 \in A_2$ and any element $a_1 \in A_1$ a morphism $X^{a_2} \to X^{a_2 + d(a_1)}$ such that this defines an equivalence of categories: $\langle A_0 \to \ker(A_1 \to A_2) \rangle \to \text{Isom}_{\mathbb{G}_m\text{-gerbes}/X}(X^{a_2}, X^{a_2 + d(a_1)})$.

To construct a resolution of the sheaf $\mathbb{G}_m$ we choose a contractible covering $U_i$ of $X$ such that all intersections $U_{i_1 \ldots i_r} = U_{i_1} \cap \ldots \cap U_{i_r}$ are also acyclic. We calculate the cohomology of $X$ by the Čech complex: $\oplus \Gamma(U_i, \mathbb{G}_m) \to \oplus \Gamma(U_{ij}, \mathbb{G}_m) \to \oplus \Gamma(U_{ijk}, \mathbb{G}_m) \to \oplus \Gamma(U_{ijkl}, \mathbb{G}_m)$.

Let $a_{ijk}$ be a cocycle. Define a groupoid: $\bigsqcup \mathbb{G}_m \times U_{ij} \xto{\sim} \bigsqcup U_i \xto{\sim} X^{a_{ijk}} \xto{\sim} \bigsqcup U_i \xto{\sim} X$.

Furthermore, given $b_{ij} \in \bigsqcup \Gamma(U_{ij}, \mathbb{G}_m)$ we can define a map of groupoids by: $(s_{ij}, x) \mapsto (b_{ij}s_{ij}, x)$. This is compatible with $m$ since:

$$(s_{ij}b_{ij})(s_{jk}b_{jk})(a_{ijk}b_{ijk}^{-1}b_{ik}b_{jk}^{-1}) = (s_{ij}s_{jk}a_{ijk})b_{ik}.$$  

Finally, an element $c_i \in \bigsqcup \Gamma(U_i, \mathbb{G}_m)$ defines a 2-morphism: $U_i \to \mathbb{G}_m \times U_i$ by $x \mapsto c_i c_j^{-1}$.

It is easy to check that these maps define the claimed equivalences, since any $\mathbb{G}_m$-gerbe restricted to the contractible space $U_i$ is trivial and automorphisms of the trivial gerbe are given by line bundles on the base. Thus the choice of trivializations of the restrictions of the gerbe to $U_i$ and a choice of trivialisations of the line bundles obtained from the two trivialisations on $U_{ij}$ defines a cocycle $a_{ijk}$. Similarly one checks the claim on morphisms.

It is immediate from the definitions that quasi-isomorphic complexes $A_\bullet \to B_\bullet$ define equivalent 2-categories.  

**References**


