



Differential Geometry/Group Theory

Growth of discrete groups of isometries in negative curvature: a gap-property

G rard Besson ^a, Gilles Courtois ^b, Sylvestre Gallot ^a

^a Institut Fourier, universit  de Grenoble I, UMR 5582 CNRS-UJF, 38402 Saint-Martin-d'H res, France

^b  cole polytechnique, centre de math matiques, UMR 7640 du CNRS, 91128 Palaiseau cedex, France

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Abstract

We prove that a finitely generated group acting without fixed point on a n -dimensional Cartan–Hadamard manifold of pinched sectional curvature $-a^2 \leq K \leq -1$ is either virtually nilpotent or has entropy $\text{Ent}(\Gamma) \geq C(n, a) > 0$. **To cite this article:** G. Besson et al., *C. R. Acad. Sci. Paris, Ser. I 341 (2005)*.

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R sum 

Croissance des groupes discrets d'isom tries en courbure strictement n gative : un minorant universel. Nous prouvons qu'un sous groupe de type fini Γ , non virtuellement nilpotent, du groupe des isom tries d'une vari t  de Cartan–Hadamard de dimension n et de courbure sectionnelle v rifiant $-a^2 \leq K \leq -1$ est d'entropie alg brique minor e, $\text{Ent}(\Gamma) \geq C(n, a) > 0$. **Pour citer cet article :** G. Besson et al., *C. R. Acad. Sci. Paris, Ser. I 341 (2005)*.

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Version fran aise abr g e

Soit Γ un groupe de type fini.   chaque syst me g n rateur fini S de Γ , on associe la distance des mots (not e d_S) et l'entropie associ e, not e $\text{Ent}_S(\Gamma)$.

Le groupe Γ est dit   croissance exponentielle si, pour au moins un (et donc pour tout) syst me g n rateur S , on a $\text{Ent}_S(\Gamma) > 0$. Il est dit   croissance exponentielle uniforme si $\text{Ent}(\Gamma) := \inf_S \{\text{Ent}_S(\Gamma)\} > 0$ ou, de mani re  quivalente, si $\text{Ent}_S(\Gamma) \geq C > 0$, o  C est une constante ne d pendant que de Γ .

Gromov a pos  la question de savoir si la croissance exponentielle implique la croissance exponentielle uniforme. Bien que la r ponse soit n gative en g n ral [15], elle est positive pour les groupes lin aires [10,4], pour les groupes r solubles [14], pour les groupes hyperboliques [13] et relativement hyperboliques [16], et pour les groupes d'iso-

E-mail addresses: G.Besson@fourier.ujf-grenoble.fr (G. Besson), courtois@math.polytechnique.fr (G. Courtois), Sylvestre.Gallot@fourier.ujf-grenoble.fr (S. Gallot).

métries discrets et géométriquement finis des variétés de Cartan–Hadamard [1]. Dans cette note nous établissons une minoration universelle de $\text{Ent}(\Gamma)$ par une constante $C > 0$ indépendante de Γ , c'est le

Théorème principal. *Soit (X^n, g) une variété de Cartan–Hadamard dont la courbure sectionnelle vérifie $-a^2 \leq K_g \leq -1$. Il existe une constante $C(n, a) > 0$ (ne dépendant que de n et de a , voir (1)) telle que, si Γ est un groupe discret de type fini qui agit sur X^n par isométries et sans point fixe alors soit Γ est virtuellement nilpotent, soit $\text{Ent}(\Gamma) \geq C(n, a)$.*

Signalons que, dans une communication privée, M. Kapovitch nous a dit avoir un résultat analogue (par une preuve complètement différente) et nous a communiqué un schéma de preuve [12] et que E. Breuillard et T. Gelander ont récemment trouvé un résultat similaire pour les sous-groupes de $GL(d, K)$ lorsque d est fixé (cf. [5]).

Schéma de preuve. Supposons Γ non virtuellement nilpotent. De tout système générateur fini S , on peut extraire deux éléments σ_1, σ_2 qui engendrent un sous-groupe Λ non virtuellement nilpotent. Le lemme de Margulis donne alors une constante $\mu(n, a) > 0$ telle que

$$L := \inf_{x \in X^n} (\text{Max}\{\rho(x, \sigma_1 x), \rho(x, \sigma_2 x)\}) \geq \mu(n, a).$$

Il nous suffit de prouver que $\text{Ent}_\Sigma(\Lambda) \geq C(n, a)$, où $\Sigma = \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$.

Deux cas se présentent : soit il existe un élément hyperbolique de longueur algébrique inférieure ou égale à 4 et de déplacement « grand », et on conclut alors par un argument de type Ping-Pong « quantifié » (cf. [3]), soit tous les éléments de longueur algébrique inférieure ou égale à 4 ont un « petit » déplacement, et alors on construit, pour tout $c > \text{Ent}_\Sigma(\Lambda) + L - \eta$ (où $\eta = \eta(n, a)$ est une constante universelle), une application c -lipschitzienne du graphe de Cayley de Λ dans X^n , ce qui nous permet de minorer $\text{Ent}_\Sigma(\Lambda)$ par $\eta(n, a)$.

1. Introduction

Let Γ be a finitely generated group. For a finite set of generators S , the word length $l_S(\gamma)$ of an element $\gamma \in \Gamma$ is defined as the smallest integer $n \geq 0$ such that there exist $s_1, \dots, s_n \in S \cup S^{-1}$ with $\gamma = s_1 s_2 \cdots s_n$. This gives rise to a left invariant distance d_S on Γ defined by $d_S(\gamma_1, \gamma_2) = l_S(\gamma_2^{-1} \gamma_1)$. The entropy of Γ associated to the set of generators S , is defined as

$$\text{Ent}_S(\Gamma) = \lim_{n \rightarrow \infty} n^{-1} \log(\#\{\gamma \in \Gamma; d_S(e, \gamma) \leq n\}),$$

and the algebraic entropy of Γ as $\text{Ent}(\Gamma) = \inf_S \{\text{Ent}_S(\Gamma)\}$, where S runs through the set of finite generating subsets of Γ .

A finitely generated group Γ is said to have exponential growth if, for some (and hence for every) finite generating subset S , $\text{Ent}_S(\Gamma) > 0$, and to have uniform exponential growth if $\text{Ent}(\Gamma) > 0$.

In 1980, Gromov raised the following question: *does exponential growth imply uniform exponential growth?* (cf. [11, Remark 5.12]).

Wilson recently answered this question negatively (cf. [15]). Nevertheless, there are classes of groups for which the answer is positive, for example hyperbolic groups (Koubi [13]), relatively hyperbolic groups (Xiangdong Xie [16]), non-elementary geometrically finite groups acting on a Cartan–Hadamard manifold (Alperin and Noskov [1]), solvable groups (Osin [14]); to complete this list see the survey [8]. In 2005, Eskin, Mozes and Oh (cf. [10]) proved that the answer is also positive for finitely generated linear groups over a field of zero characteristic: more precisely, if such a group Γ is not virtually nilpotent, it has uniform exponential growth. This was recently generalized to finitely generated linear groups with any characteristic by Breuillard and Gelander [4].

In the above results, the inequality which is proved writes: $\text{Ent}(\Gamma) \geq C > 0$, where the constant C depends on Γ . This allows the existence of sequences $(\Gamma_n)_{n \in \mathbf{R}}$ of groups such that $\text{Ent}(\Gamma_n)$ goes to 0, though each Γ_n has uniform exponential growth (cf. [9] for examples of such sequences of groups).

On the other hand, in view of geometric applications (such applications are given in [3]), we need a stronger gap-result, i.e. an inequality of the type: for every Γ in a class of groups, $\text{Ent}(\Gamma) \geq C > 0$, where C is an explicit universal constant which does not depend on Γ . Such a gap-result is given by the

Main Theorem. *Let (X^n, g) be a Cartan–Hadamard manifold with sectional curvature $-a^2 \leq K_g \leq -1$. There exists a positive number $C(n, a)$ (depending only on n and a) such that, if Γ is a finitely generated discrete group of isometries acting on X^n without fixed point, then either Γ is virtually nilpotent or $\text{Ent}(\Gamma) \geq C(n, a)$.*

In particular, the theorem says that any such group Γ either has zero entropy or has its entropy bounded from below by a uniform constant.

The exact value of the universal constant is

$$C(n, a) = \text{Min} \left(\frac{\log(\cosh(\mu/4))}{4 + \log(\cosh(\mu/4))} \cdot \frac{\log 2}{6}, 10^{-3} \left(1 - \frac{\cosh(\mu/4)}{\cosh(\mu/2)} \right)^4 \right) \tag{1}$$

where $\mu = \mu(n, a)$ is the Margulis constant of (X^n, g) (see the definition in [6, p. 281]).

In a private communication, M. Kapovitch told us that he has a personal (and completely different) proof of an analogous result and gave us a sketch of this proof [12].

Breuillard and Gelander recently prove a similar gap-result when Γ is a discrete finitely generated subgroup of $GL(d, K)$ (where K is a local field of any characteristic): namely either Γ is virtually nilpotent or $\text{Ent}(\Gamma) \geq C(d)$, where $C(d)$ is some constant which only depends on d (cf. [5]).

Notation. For a Cartan–Hadamard manifold (X^n, g) , let us denote by ρ the distance associated to the Riemannian metric g . For an isometry γ of (X^n, g) , the displacement of γ is defined by $l(\gamma) = \inf_{x \in X^n} \rho(x, \gamma x)$. For two isometries σ_1 and σ_2 , let us define the number $L(\sigma_1, \sigma_2)$ as $\inf_{x \in X^n} (\text{Max}\{\rho(x, \sigma_1 x), \rho(x, \sigma_2 x)\})$.

2. Sketch of proof

Consider a finite generating set S of Γ ; if Γ is not virtually nilpotent, there exist two elements in S , say σ_1, σ_2 , such that the group $\langle \sigma_1, \sigma_2 \rangle$ generated by σ_1 and σ_2 is not virtually nilpotent and therefore, by the Margulis lemma [6, p. 281], $L(\sigma_1, \sigma_2) \geq \mu(n, a)$.

We now consider the subgroup Λ generated by $\Sigma = \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$ and show that $\text{Ent}_\Sigma(\Lambda) \geq C(n, a)$. The proof is then divided in two cases: either there exists an element γ of bounded word length (say $l_\Sigma(\gamma) \leq 4$) whose displacement $l(\gamma)$ is ‘big’ or every such element has ‘small’ displacement. In the first case we use a Ping-Pong argument to find a free semi-group at distance bounded by some universal constant $D(n, a)$. In the second case, we compute a universal constant $\eta = \eta(n, a)$ and, for every $c > \text{Ent}_\Sigma(\Lambda) + L(\sigma_1, \sigma_2) - \eta$, we construct a Γ -equivariant Lipschitz map (with Lipschitz constant c) from the Cayley graph of Λ to X^n . As $c \geq L(\sigma_1, \sigma_2)$ by the definition of $L(\sigma_1, \sigma_2)$, we obtain $\text{Ent}_\Sigma(\Lambda) \geq \eta(n, a)$.

3. Some details

We start with the following lemma:

Lemma 3.1. *Let Γ be as above and moreover non-virtually nilpotent. For any finite generating system $S = \{\sigma_1, \dots, \sigma_k, \sigma_1^{-1}, \dots, \sigma_k^{-1}\}$, there exists i, j ($i \neq j$) such that the group $\langle \sigma_i, \sigma_j \rangle$ generated by σ_i and σ_j is not virtually nilpotent and $L(\sigma_i, \sigma_j) \geq \mu(n, a)$.*

Proof. The proof of the first statement is done by contradiction. Precisely, if it is not true, then for all $1 \leq i, j \leq k$ the subgroup generated by σ_i and σ_j is virtually nilpotent. Under our assumptions, discrete virtually nilpotent subgroups of (X^n, g) are either virtually Abelian preserving a geodesic line or fix a point at infinity and preserve all horospheres centered at this point. One easily deduces that Γ has the same behaviour and thus is virtually nilpotent. The inequality $L(\sigma_i, \sigma_j) \geq \mu(n, a)$ follows from the Margulis lemma, see [6, p. 281]. \square

For the sake of simplicity let us assume that $(i, j) = (1, 2)$ in Lemma 3.1. Let Λ be the discrete group generated by σ_1 and σ_2 endowed with the generating system $\Sigma = \{\sigma_1, \sigma_2, \sigma_1^{-1}, \sigma_2^{-1}\}$. We denote by $B_\Sigma(N)$ the ball of radius N in Λ with respect to the word metric d_Σ , centered at the neutral element e .

Lemma 3.2. *Let us assume that there exists $\gamma \in B_\Sigma(4)$ such that the displacement $l(\gamma)$ satisfies $l(\gamma) \geq \delta > 0$ (for some positive number δ). Then there exists $i \in \{1, 2\}$, such that, for $N = \lceil \frac{4}{\delta} \rceil + 1$, one of the two semi-groups generated by $\{\gamma^N, (\sigma_i \gamma \sigma_i^{-1})^N\}$ or $\{\gamma^N, (\sigma_i \gamma \sigma_i^{-1})^{-N}\}$ is free. Therefore $\text{Ent}_\Sigma(\Lambda) \geq \frac{\log 2}{6N}$.*

For the proof of Lemma 3.2, see [3]; this proof is similar to the classical Ping-Pong argument (cf. [7, p. 188]), except for the fact that our result computes N independently from the group Γ .

In Lemma 3.2, δ is arbitrary and, later on, we shall choose it in terms of L . Lemma 3.2 will then solve the first case of the sketch of proof, namely the case where there is a ‘big’ hyperbolic element in $B_\Sigma(4)$.

Let us now consider the second case, namely the case where the displacement $l(\gamma)$ of every $\gamma \in B_\Sigma(4) - \{e\}$ is less than δ . Let us denote by $G = G(\Lambda, \Sigma)$ the Cayley graph of Λ , endowed with the generating system Σ , and by d_G the distance on G such that the edges have length one and which coincides with the word distance d_Σ on the vertices. For the sake of simplicity, let us denote $L(\sigma_1, \sigma_2)$ by L . We first construct a Lipschitz equivariant map $F_c : G \rightarrow X^n$ with Lipschitz constant $c > 0$. For any such map, one has:

$$L \leq \text{Max}_{i=1,2} \{ \rho(F_c(e), \sigma_i F_c(e)) \} = \text{Max}_{i=1,2} \{ \rho(F_c(e), F_c(\sigma_i)) \} \leq c \text{Max}_{i=1,2} \{ d_G(e, \sigma_i) \} = c. \tag{2}$$

For the sake of simplicity let us assume that there exists a point $x_0 \in X^n$ where the function $x \mapsto \text{Max}_{i=1,2} \{ \rho(x, \sigma_i x) \}$ achieves its minimum L (in the general case, we approximate L by replacing x_0 by a minimizing sequence of points). For every $y \in X^n$ and $s \in G = G(\Lambda, \Sigma)$, if c is a positive real number and μ some compactly supported smooth measure on X^n , let us consider the measure

$$\mu_s^c = \sum_{\gamma \in \Lambda} e^{-c \cdot d_G(s, \gamma)} \gamma_* \mu$$

and let us define the function P_c by

$$P_c(y, s) = \int_X \cosh[\rho(y, z)] d\mu_s^c(z).$$

Roughly speaking, μ and $P_c(y, s)$ may be seen as smooth approximations of δ_{x_0} and $\sum_{\gamma \in \Lambda} e^{-c \cdot d_G(s, \gamma)} \cosh[\rho(y, \gamma x_0)]$, respectively. For each c such that P_c converges, the function $y \rightarrow P_c(y, s)$ is strictly convex (and moreover C^2) and goes to infinity with y . We thus define $F_c(s)$ to be the unique point where this function achieves its minimum.

Lemma 3.3. *The map F_c is equivariant and Lipschitz with Lipschitz constant c .*

Sketch of proof. The map F_c is Lipschitz and DF_c can be computed by the implicit function theorem at any point s which is neither a vertex nor the middle of an edge. One can then check that $\|DF_c\| \leq c$ at s , this estimate coming from the fact that the sectional curvature of (X^n, g) is bounded above by -1 , and thus that $\text{Hess}(\cosh(\rho)) \geq \cosh(\rho) \cdot g$ by the Rauch’s comparison theorem, see details in [2]. \square

Let us recall that $L = \text{Max}\{\rho(x_0, \sigma_1 x_0), \rho(x_0, \sigma_2 x_0)\}$; thus $\rho(x_0, \gamma x_0) \leq L \cdot l_\Sigma(\gamma)$ by the triangle inequality and it is easy to deduce that, for every y and s , the series $P_c(y, s)$ converges for every $c > \text{Ent}_\Sigma(\Lambda) + L$. However, we have to improve this estimate in such a way that this series will still converge for every $c > \text{Ent}_\Sigma(\Lambda) + L - \eta(n, a)$ for some positive universal constant $\eta(n, a)$. In order to prove this, let us settle the:

Definition 3.4. Let $\eta > 0$ be a positive number and γ an element of Λ . We say that γ is η -straight if $\rho(x_0, \gamma x_0) > (L - \eta)l_\Sigma(\gamma)$.

This definition depends on the choice of x_0 made before.

Lemma 3.5. *Let $\delta = \log(\cosh(L/4))$ and $\eta = 10^{-3}(1 - \frac{\cosh(L/4)}{\cosh(L/2)})^4$. If, for every $\gamma \in B_\Sigma(4) - \{e\}$, one has $l(\gamma) < \delta$, then*

- (i) None of the γ 's such that $l_\Sigma(\gamma) = 6$ is η -straight,
- (ii) For every $\gamma \in \Lambda$, we have $\rho(x_0, \gamma x_0) \leq (L - \eta) d_G(e, \gamma) + 5\eta$.

Sketch of proof. As (ii) is an obvious consequence of (i), we only have to prove (i). By easy computations, one can verify that any reduced word of length 6 contains a subword of the type α^2 or of the type $\alpha\beta\alpha$, where α and β have length 1 or 2. The part (i) then follows from the triangle inequality and from the

Lemma 3.6. *Under the assumptions of Lemma 3.5, for any α and β of length 1 or 2, then α^2 (resp. $\alpha\beta\alpha$) cannot be $(C\eta)$ -straight with $C = 3$ (resp. $C = 2$).*

Lemma 3.6 is a corollary of the following geometric lemmas that we do not prove here (they can be deduced from the strong convexity of the distance function ρ).

Lemma 3.7. *Let X^n be a Cartan–Hadamard manifold with sectional curvature $K \leq -1$. Let δ' and L' be two positive numbers such that $L' > \text{Argcosh}(e^{\delta'})$. For any isometry γ of X^n such that $l(\gamma) < \delta'$ and for any $x_0 \in X^n$ such that $\rho(x_0, \gamma x_0) \geq L'$ we have $\rho(x_0, \gamma^2 x_0) < 2\rho(x_0, \gamma x_0) - (1 - \frac{e^{\delta'}}{\cosh(L')})^2$.*

Lemma 3.8. *Let X^n be a Cartan–Hadamard manifold with sectional curvature $K \leq -1$. Let us consider four points y_0, y_1, y_2, y_3 such that $\rho(y_0, y_1) + \rho(y_1, y_2) - \rho(y_0, y_2) \leq \eta_1$ and $\rho(y_1, y_2) + \rho(y_2, y_3) - \rho(y_1, y_3) \leq \eta_2$, then $\rho(y_0, y_1) + \rho(y_1, y_2) + \rho(y_2, y_3) - \rho(y_0, y_3) \leq (1 + \frac{\rho(y_2, y_3)}{\rho(y_1, y_2)})(\eta_1 + \text{Argcosh}(e^{\eta_2}))$.*

3.1. Sketch of proof of Lemma 3.6

Case α^2 : If α^2 is 3η -straight, then so is α (because α^2 is a reduced word) and thus, $\rho(x_0, \alpha x_0) \geq (L - 3\eta)l_\Sigma(\alpha)$. But $l_\Sigma(\alpha) \leq 2$, thus, by assumption, the displacement of α satisfies $l(\alpha) \leq \delta$ and therefore the geometric Lemma 3.7, applied to $\gamma = \alpha$, $\delta' = \delta$ and $L' = L/2$, implies that $\rho(x_0, \alpha^2 x_0) < 2\rho(x_0, \alpha x_0) - (1 - \frac{e^\delta}{\cosh(L/2)})^2$, which contradicts the fact that α^2 is 3η -straight.

Case $\alpha\beta\alpha$: Let us choose $y_0 = x_0$, $y_1 = (\alpha\beta)x_0$, $y_2 = (\alpha\beta\alpha)x_0$ and $y_3 = (\alpha\beta)^2 x_0$. If $\alpha\beta\alpha$ is (2η) -straight, then one can show that $\alpha\beta$ is $(C'\eta)$ -straight, where $C' = 2\frac{l_\Sigma(\alpha\beta\alpha)}{l_\Sigma(\alpha\beta)}$ and that α is (4η) -straight. By the triangle inequality and Lemma 3.8 (applied with $\eta_1 = \eta_2 = 12\eta$), we get:

$$2\rho(x_0, \alpha\beta x_0) - \rho(x_0, (\alpha\beta)^2 x_0) \leq \left(1 - \frac{e^\delta}{\cosh(L/2)}\right)^2.$$

This inequality contradicts the geometrical Lemma 3.7 (this lemma applies here because $l_\Sigma(\alpha\beta) \leq 4$ implies $l(\alpha\beta) < \delta$ by assumption and $\rho(x_0, \alpha\beta x_0) \geq 2(L - C'\eta) \geq L$). \square

Let us now finish the proof of the Main Theorem in the second case, namely the case where every γ in $B_\Sigma(4)$ has displacement less than $\delta = \log(\cosh(L/4))$. By Lemma 3.5(ii), one can see that the series $P_c(x_0, e)$, and hence all the $P_c(y, s)$'s, converges for every $c > \text{Ent}_\Sigma(\Lambda) + L - \eta$. As we have seen in Eq. (2) that $c \geq L$, we immediately deduce that $\text{Ent}_\Sigma(\Lambda) \geq \eta$.

Summarizing the above estimates we obtain $\text{Ent}_S(\Gamma) \geq \text{Ent}_\Sigma(\Lambda) \geq \text{Min}\{\frac{\delta}{4+\delta} \frac{\log 2}{6}, \eta\}$. To end the proof, let us notice that the last inequality still gives a lower bound of $\text{Ent}_S(\Gamma)$ when one replaces L by $\mu(n, a)$ in the expressions of δ and η (this comes from the fact that $L \geq \mu(n, a)$ by Lemma 3.1 and from the fact that δ and η are increasing functions of L).

References

- [1] R.C. Alperin, G.A. Noskov, Non vanishing of algebraic entropy for geometrically finite groups of isometries of Hadamard manifolds, Preprint, math.GR/0408160.
- [2] G. Besson, G. Courtois, S. Gallot, Rigidity of amalgamated products in negative curvature, Preprint, math.DG/0506350.
- [3] G. Besson, G. Courtois, S. Gallot, Un lemme de Margulis sans courbure et ses applications, Prépublication Inst. Fourier 595 (2003) 1–59.
- [4] E. Breuillard, T. Gelander, An effective Tits alternative, Abstract, <http://www.dma.ens.fr/~breuilla/>.

- [5] E. Breuillard, T. Gelander, Cheeger constant and algebraic entropy of linear groups, Preprint, June 2005.
- [6] Y.D. Burago, V.A. Zalgaller, Geometric Inequalities, Grundlehren Math. Wiss., vol. 285, Springer-Verlag, Berlin/New York, 1988.
- [7] P. De la Harpe, Topics in Geometric Group Theory, Chicago Lectures in Math., Univ. of Chicago Press, Chicago, 2000.
- [8] P. De la Harpe, Uniform growth in groups of exponential growth, *Geom. Dedicata* 95 (2002) 1–17.
- [9] P. De la Harpe, R. Grigorchuk, Limit behaviour of exponential growth rates for finitely generated groups, *Monographie de l'Enseignement Mathématique* 38 (2001) 351–370.
- [10] A. Eskin, S. Mozes, H. Oh, On uniform exponential growth for linear groups, *Invent. Math.* 160 (2005) 1–30.
- [11] M. Gromov, in: J. Lafontaine, P. Pansu (Eds.), *Metric Structures for Riemannian and Non-Riemannian Spaces*, in: *Progress in Math.*, Birkhäuser, Basel, 1999.
- [12] M. Kapovitch, On Ping–Pong argument, Private communication, 2003 July 26th.
- [13] M. Koubi, Croissance uniforme dans les groupes hyperboliques, *Ann. Inst. Fourier* 48 (1998) 1441–1453.
- [14] D. Osin, The entropy of solvable groups, *Ergodic Theory Dynam. Systems* 23 (2003) 907–918.
- [15] J. Wilson, On exponential growth and uniformly exponential growth for groups, *Invent. Math.* 155 (2004) 287–303.
- [16] X. Xie, Growth of relatively hyperbolic groups, Preprint, math.GR/0504191.