Partial Differential Equations

Application of global Carleman estimates with rotated weights to an inverse problem for the wave equation

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Abstract

We establish geometrical conditions for the inverse problem of determining a stationary potential in the wave equation with Dirichlet data from a Neumann measurement on a suitable part of the boundary. We present the stability results when we measure on a part of the boundary satisfying a rotated exit condition. The proofs rely on global Carleman estimates with angle type dependence in the weight functions. To cite this article: A. Doubova, A. Osses, C. R. Acad. Sci. Paris, Ser. I 341 (2005). © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

Résumé


Version française abrégée

Soient $\Omega$ un ouvert borné de $\mathbb{R}^2$ de frontière $\partial\Omega$ de classe $C^2$, $\Gamma_0$ une partie ouverte non vide de $\partial\Omega$, $\nu$ la normale unitaire extérieure à $\Omega$, $T > 0$ et $U_M = \{q \in L^\infty(\Omega); \|q\|_{L^\infty(\Omega)} \leq M\}$, où $M > 0$ est une constante. On s’intéresse à la stabilité du problème inverse suivant : \textit{Etant donné $u_0$, $u_1$, $h$ et $\xi$ dans des espaces appropriés, on cherche des conditions suffisantes sur $\Gamma_0$ et $T$ telles qu’il existe $q \in U_M$ tel que la solution $u$ de l’équation des ondes}

\[
\begin{align*}
\partial_{tt} u - \Delta u + q(x)u &= 0 & \text{dans } \Omega \times (0, T), \\
u u &= h & \text{sur } \partial \Omega \times (0, T), \\
u(u(x, 0) &= u_0, \quad \partial_t u(x, 0) = u_1 & \text{dans } \Omega}
\end{align*}
\]

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La constante de stabilité dans (6) ne dépend que de

\( q \in \mathbb{R}^2 \), respectivement et avec des mesures \( \xi \) données par (2) et \( \bar{\xi} = \frac{\partial q}{\partial \nu} \) Il existe un temps \( T_0 = T_1(\Omega, \theta, x_0) > 0 \) tel que si \( T > T_1 \), \( M > 0 \) et \( |\bar{u}(x, 0)| \geq \alpha_0 > 0 \) presque partout dans \( \Omega \), alors il existe une constante \( C = C(T, M, M_1, \alpha_0) > 0 \) telle que

\[
\| q - \bar{q} \|_{L^2(\Omega)} \leq C \| \xi - \bar{\xi} \|_{H^1(\Omega \times T; L^2(\Gamma_0))} \quad \forall q \in \mathcal{U}_M.
\]


1. Introduction and main results

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^2 \) with boundary \( \partial \Omega \) of class \( C^2 \), \( \Gamma_0 \) a nonempty open subset of \( \partial \Omega \), \( \nu \) the unit outward normal vector to \( \Omega \), \( T > 0 \) and \( \mathcal{U}_M = \{ q \in L^\infty(\Omega) : \| q \|_{L^\infty(\Omega)} \leq M \} \), where \( M > 0 \) is a constant. We are interested in the stability of the following inverse problem: Given \( u_0, u_1, h \) and \( \xi \) in appropriate spaces, we look for sufficient conditions under \( \Gamma_0 \) and \( T \) such that we can find \( q \in \mathcal{U}_M \) such that the solution \( u \) of the wave equation

\[
\begin{align*}
\partial_t u - \Delta u + q(x) u &= 0 \quad &\text{in } \Omega \times (0, T), \\
u u &= h \quad &\text{on } \partial \Omega \times (0, T), \\
u(x, 0) &= u_0, \quad \partial_t u(x, 0) = u_1 \quad &\text{in } \Omega
\end{align*}
\]
satisfies the additional condition
\[
\frac{\partial u}{\partial y} = \xi \quad \text{on } \Gamma_0 \times (0, T).
\]  

(8)

Let us denote by \( u \) and \( \bar{u} \) the solutions of (7) associated to \( q \) and to some known potential \( \bar{q} \) respectively. If we set \( f = q - \bar{q} \) and \( y = u - \bar{u} \) then \( q u - \bar{q} \bar{u} = q y + f \bar{u} \) and we obtain the following equivalent inverse problem for the function \( y \): Given \( q \in \mathcal{U}_M, R = -\bar{u} \) and \( \bar{\xi} = \frac{\partial \bar{u}}{\partial \nu} \) in appropriate spaces, we look for sufficient conditions for \( \Gamma_0 \) and \( T \) such that we can find a time independent function \( f \) such that the solution \( y \) of
\[
\left\{ \begin{array}{lcr}
\frac{\partial y}{\partial t} - \Delta y + q(x) y &=& f(x) R(x, t) \quad \text{in } \Omega \times (0, T), \\
y(0, t) &=& 0 \quad \text{on } \partial \Omega, \\
y(x, 0) &=& \partial_t y(x, 0) = 0 \quad \text{in } \Omega
\end{array} \right.
\]
satisfies the additional condition
\[
\frac{\partial y}{\partial y} = \xi - \bar{\xi} \quad \text{on } \Gamma_0 \times (0, T).
\]  

(9)

Let \( a > 0, b \in \mathbb{R} \) with \( a^2 + b^2 = 1 \) and
\[
\theta = \tan^{-1}(b/a) \in (-\pi/2, \pi/2), \quad a = \cos \theta, \ b = \sin \theta.
\]  

(10)

The main novelty here is that we can consider \( \Gamma_0 \) sufficiently large in such a way that
\[
\exists x_0 \in \mathbb{R}^2 \setminus \bar{\Omega} \quad \text{such that} \quad \Gamma_0 \supseteq \{ x \in \partial \Omega : (x - x_0) \cdot (aI - bA) \nu(x) > 0 \},
\]  

(12)

where \( A = (a_{ij}), i, j \in \{1, 2\}, a_{11} = a_{22} = 0, a_{12} = -1, a_{21} = 1 \) and \( I \) is the identity matrix. Notice that \((aI - bA)\nu(x)\) corresponds to a clockwise rotation of the normal field in an angle \( \theta \). In [9] and [5] the stability results were obtained when \( \Gamma_0 \) includes the exit boundary region with respect to some exterior point \( x_0 \) (see Fig. 1 left). In this Note we extend this geometrical hypothesis in the sense that it suffices that \( \Gamma_0 \) contains a rotated exit part of the boundary which can be described by the angle parameter \( \theta \). For \( \theta = 0 \) we recover the exit boundary region (see Fig. 1 left). For \( \theta \to \pi/2 \) and \( \theta \to -\pi/2 \) we obtain for example more general geometrical conditions (see Fig. 1 center). There are of course an infinite number of intermediate cases for \( \theta \in (-\pi/2, \pi/2) \). This geometrical condition is only sufficient in order to solve the inverse problem. A reasonable conjecture is that it is only necessary to measure in order to capture the geometrical optic rays (see [1]) coming from the support of the unknown potential. In fact, a generic stability can be obtained (see [10]) for the same inverse problem considered in this Note under this weaker condition.

\textbf{Theorem 1.1.} Suppose that \( \Gamma_0 \) satisfies (12). Let \( u \) and \( \bar{u} \) be the respective solutions of (7) associated to \( q \in \mathcal{U}_M \) and \( \bar{q} \in L^\infty(\Omega) \) with measurements \( \xi \) given by (8) and \( \bar{\xi} = \frac{\partial \bar{u}}{\partial \nu} \). There exists a time \( T_1 = T_1(\Omega, \theta, x_0) > 0 \) such that if \( T > T_1, \bar{u} \in H^1(0, T; L^\infty(\Omega)) \) and \( |\tilde{u}(x, 0)| \geq \alpha_0 > 0 \) a.e. in \( \Omega \), then there exists a constant \( C = C(T, M, \|\bar{q}\|_{L^\infty(\Omega)}, \|\tilde{u}\|_{H^1(L^\infty)}, \alpha_0) > 0 \) such that
\[
\|\tilde{q} - q\|_{L^2(\Omega)} \leq C\|\xi - \bar{\xi}\|_{H^1(0, T; L^2(\Gamma_0))} \quad \forall q \in \mathcal{U}_M.
\]  

(13)

Fig. 1. From left to right: rotated exit boundary regions in a domain corresponding to \( \theta = 0, \theta \to \pi/2 \) and \( \theta \to -\pi/2 \) with respect to some exterior point \( x_0 \). The last figure shows the main geometrical parameters of the problem.

Fig. 1. De gauche à droite : les régions de la frontière satisfaisant une condition sortante à directions variables correspondante à \( \theta = 0, \theta \to \pi/2 \) et \( \theta \to -\pi/2 \) par rapport à un point extérieur \( x_0 \). La dernière figure montre les principaux paramètres géométriques du problème.
Notice that this result is of a local nature, since the stability holds around some $\tilde{q} \in L^\infty(\Omega)$ for which the corresponding solution $\tilde{u}$ of problem (7) is regular enough. The proof of the Theorem 1.1, in view of the previous discussions, is a particular case of the following result. It is sufficient to take $f = q - \tilde{q}$ and to use the fact that the stability constant in (14) depends only on $M$.

**Theorem 1.2.** Assume $\Gamma_0$ and $T$ verify the hypothesis of Theorem 1.1. Let be given $f \in L^2(\Omega)$, $R \in H^1(0, T; L^\infty(\Omega))$ with $\|\partial_t R\|_{L^2(\Omega)} \leq M_1$ and $|R(x, 0)| \geq \alpha_0 > 0$ a.e. in $\Omega$. Then, there exists a constant $C = C(T, M, M_1, \alpha_0) > 0$ such that if $y$ is the solution of (9) we have
\[
\|f\|_{L^2(\Omega)} \leq C \left\| \frac{\partial y}{\partial \nu} \right\|_{H^1(0, T; L^2(\Gamma_0))} \forall q \in \mathcal{U}_M.
\] (14)

The proof of this theorem will be sketched in Section 3. It is based on the obtainment of a global Carleman estimate with a modified weight given in Proposition 2.1. We are able to construct the appropriate weight function only in two dimensions. More details will be given in a forthcoming article [2], where an explicit expression for the inversion time $T_1$ is given. We make here only two remarks about this time:

**Remark 1.** As $\theta \to \pm \frac{\pi}{4}$, we have $a \to 0$ and $b \to \pm 1$ and the inversion time $T_1 \to +\infty$. More precisely
\[
T_1 \approx \exp(\bar{\delta} \|\bar{\delta}\| \tan \theta)(R_0^2 \exp(\bar{\delta} \|\bar{\delta}\| \tan \theta) - r_0^2)^{1/2},
\] (15)
\[
R_0 = \sup_{x \in \Omega} |x - x_0|, \quad r_0 = \inf_{x \in \Omega} |x - x_0|, \quad \delta \theta = \sup_{x \in \Omega} \arg(x - x_0) - \inf_{x \in \Omega} \arg(x - x_0).
\] (16)

This fact recalls the one encountered in [8] where the exact controllability time for the wave equation with a rotated boundary control region was of order $\frac{2R_0}{\cos \theta}$ as $\theta \to \pm \frac{\pi}{4}$.

**Remark 2.** In the case $\theta = 0$ we also recover $T_1 = (R_0^2 - r_0^2)^{1/2}/\sqrt{\beta}$, where the constant $\beta$ is introduced in the next section (see [5,9] and the idea for a better estimation of this time in [6]).

**Remark 3.** The hypothesis $x_0 \notin \bar{\Omega}$ in (12) is maybe not necessary in Theorems 1.1 and 1.2 (see [2]).

2. Global Carleman inequality with rotated weights

Let us set $Q = \Omega \times (-T, T)$, $\Sigma_0 = \Gamma_0 \times (-T, T)$. Given $\beta \in (0, \beta_2)$ with $\beta_2 = \beta_2(a, b, \theta, x_0)$, we introduce the following weight function:
\[
\Phi(x, t) = a\phi(x) \exp(2a \tilde{\phi}(x)/b) - \beta t^2, \quad \varphi(x, t) = \exp(\lambda \Phi(x, t)) \quad x \in \bar{\Omega}, \ t \in \mathbb{R}, \ \lambda > 0,
\] (17)
where $\phi$ and $\tilde{\phi}$ are given by
\[
\phi(x) = |x - x_0|^2, \quad \tilde{\phi}(x) = \arg(x - x_0).
\] (18)

**Proposition 2.1.** Assume that $\Gamma_0$ satisfies (12). Then for all $M > 0$, there exist positive constants $\tilde{\lambda}$, $\tilde{\mu}$ and $C$ depending on $a$, $b$, $\Omega$ and $x_0$ such that for any $q \in L^\infty(Q)$ with $\|q\|_{L^\infty(Q)} \leq M$, for any $\lambda \geq \tilde{\lambda}$ and for any $s \geq \tilde{s} = \exp(\tilde{\mu} \lambda(1 + T^2))M^{2/3}$ the following estimate holds:
\[
sl \int_Q e^{2\psi \varphi}(|\partial_t v|^2 + |\nabla v|^2) \, dx \, dt + s^3 \lambda^3 \int_Q e^{2s\lambda \varphi^3} |v|^2 \, dx \, dt \leq C \left( \int_Q e^{2s\lambda |\partial_{tt} v - \Delta v + q(x, t)v|^2} \, dx \, dt + s\lambda \int_{\Sigma_0} e^{2s\varphi} \left| \frac{\partial v}{\partial \nu} \right|^2 \, d\sigma \, dt \right)
\] (19)
for all $v \in C^2(\bar{\Omega} \times [-T, T])$ such that $v = 0$ on $\Sigma$, $v(x, \pm T) = 0$ and $\partial_t v(x, \pm T) = 0$. 
Sketch of the proof. We follow the method from [4] which involves a lot of computations not given here. Nevertheless, let us precise some important things. Let us notice first that the properties of the weight function $\Phi$ given in (17) are in narrow connection with the geometrical conditions (12) on $\Gamma$. Indeed, after splitting the corresponding conjugate operator, it appears the adjoint equation multiplied by the gradient of the solution following some direction $\nabla \Phi$. Taking the idea from [8], we choose $\Phi$ with a gradient $\nabla \Phi$ which is essentially a rotation of the original field $(x - x_0)$, but with a magnitude which radially depends in space:

$$
\nabla \Phi = (a \nabla \phi + 2b \phi \nabla \hat{\phi}) \exp(2a\hat{\phi}/b) = 2(aI + bA)(x - x_0) \exp(2a\hat{\phi}/b).
$$

Secondly, in order to have (19), the weight function $\Phi$ must satisfy in addition other algebraic conditions which are obtained with a carefully choice of $\beta_2$. □

Remark 4. Notice that in (19) the source term $q = q(x, t)$ can depend on $x$ and $t$ with a constant depending uniformly on $q$. On the other hand (19) does not require any sufficiently large size for the time interval $2T$.

3. Application to the inverse problem

Let us present the main steps of the proof of Theorem 1.2. We use the arguments of [5,9]. First, we set $\psi = \partial_t y$, where $y$ is the solution of (9). Then, we have

$$
\begin{align*}
\partial_{tt} \psi - \Delta \psi + q(x) \psi &= f(x) \partial_t R(x, t) & \text{in } \Omega \times (0, T), \\
\psi &= 0 & \text{on } \partial \Omega \times (0, T), \\
\psi(x, 0) &= 0, & \partial_t \psi(x, 0) = f(x) R(x, 0) & \text{in } \Omega.
\end{align*}
$$

Thanks to the hypothesis of Theorem 1.2 on $f$, $R$ and $q$ and using the regularity properties of the wave equation (see, for example, [3]) we deduce that $\psi \in C([0, T]; H^1_2(\Omega)) \cap C^1([0, T]; L^2(\Omega)), \frac{\partial \psi}{\partial v} \in L^2(0, T; L^2(\partial \Omega))$. Thus, we obtain that $y \in C([0, T]; H^1_2(\Omega) \cap H^2(\Omega)) \cap C^1([0, T]; H^1_2(\Omega)) \cap C^2([0, T]; L^2(\Omega))$. Extending $\partial_t R$ and $\psi$ on $\Omega \times (-T, 0)$ as even functions and calling the extensions by the same symbols, we obtain that $y$ is the solution of (9) defined now in $\Omega \times (-T, T)$ with the regularity as before.

In order to apply the Carleman estimate (19), we introduce

$$
z = e^{2\psi} \chi \partial_t y,
$$

where $\chi \in C_c^\infty([-T, T]), \ 0 \leq \chi \leq 1$, is a suitable cut-off function and $\psi$ is given by (17). Notice that $z(x, \pm T) = \partial_t z(x, \pm T) = 0$ and then after some computations we get

$$
\frac{1}{2} \int_\Omega |f(x)|^2 |R(x, 0)|^2 e^{2\psi(x, 0)} \, dx = \int_{-T}^T \int_\Omega (\partial_{tt} z - \Delta z + q(x)z) \partial_t z \, dx \, dr.
$$

(21)

In the last step, we use Carleman inequality (19) in order to estimate the right-hand side of (21) in terms of $\|\frac{\partial \psi}{\partial v}\|_{H^1(0, T; L^2(\Gamma_0))}$. At this point, we also use that $\psi(x, \pm T) < \psi(x, 0)$ uniformly for all $x \in \overline{\Omega}$ whenever $T > T_1$ and that $|R(x, 0)| \geq a_0 > 0$. This gives (14) and to conclude the proof of Theorem 1.2.

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