Abstract

We consider continued fractions

\[-\frac{a_1}{1} - \frac{a_2}{1} - \frac{a_3}{1} - \ldots\]  
(CF)

with real coefficients \(a_i\) converging to a limit \(a\). Ramanujan claimed that if \(a \neq \frac{1}{4}\), then the fraction converges if and only if \(a < \frac{1}{4}\). The statement of convergence was proved by Van Vleck in 1904 for complex \(a_i\) converging to \(a \in \mathbb{C} \setminus \left[\frac{1}{4}, +\infty\right)\). Gill proved the divergence of (CF) under the assumption that \(a_i \to a > \frac{1}{4}\) fast enough, more precisely, whenever \(\sum_i |a_i - a| < \infty\).

The Ramanujan’s conjecture saying that (CF) always diverges whenever \(a_i \to a > \frac{1}{4}\) remained, up to now, an open question. In the present Note we disprove it. We show that for any \(a > \frac{1}{4}\) there exists a real sequence \(a_i \to a\) such that (CF) converges. Moreover, we show that Gill’s sufficient divergence condition is the optimal condition on the speed of convergence of the \(a_i\’s\). To cite this article: A.A. Glutsyuk, C. R. Acad. Sci. Paris, Ser. I 341 (2005).

Résumé

Sur convergence des fractions continues généralisées et une conjecture de Ramanujan. Nous considérons une fraction continue

\[-\frac{-a_1}{1} - \frac{-a_2}{1} - \frac{-a_3}{1} - \ldots\]  
(FC)

à coefficients réels \(a_i \to a\). Ramanujan a affirmé, que si \(a \neq \frac{1}{4}\), alors la fraction converge, si et seulement si \(a < \frac{1}{4}\). La convergence a été démontrée par Van Vleck en 1904 pour \(a_i\) complexes convergeant vers un \(a \in \mathbb{C} \setminus \left[\frac{1}{4}, +\infty\right)\). Gill a démontré (en 1973), que la fraction diverge, si \(a_i \to a > \frac{1}{4}\) assez vite, plus précisément, si \(\sum_i |a_i - a| < \infty\).

La conjecture de Ramanujan disant que la fraction diverge toujours, quand \(a > \frac{1}{4}\), restait ouverte jusqu’au présent. Nous montrons, qu’elle est fausse : pour tout \(a > \frac{1}{4}\) il existe une suite réelle \(a_i \to a\) telle que la fraction converge. Nous montrons aussi,

Version française abrégée

Théorème 0.1. Pour tout $a > \frac{1}{4}$ il existe une suite $a_i \to a$, telle que la fraction (1) converge.

Théorème 0.2. Étant donnés $q \in \mathbb{N}$, $q \geq 3$, et une suite $r_i \to 0$, $r_i > 0$, telle que $\sum_i r_i = \infty$. Alors il existe toujours un $a > \frac{1}{4}$ et une suite

$$a_i \to a, \quad a_i = a, \quad si \ i \not\equiv 1, 2 (\mod q), \quad a_{qi+1} - a = O(r_i), \quad a_{qi+2} - a = O(r_i), \quad quand \ i \to \infty,$$

telle que la fraction (1) converge.

1. Main results and the plan of the paper

We consider continued fractions

$$-a_1 \quad \frac{-a_2}{1 - \frac{-a_3}{1 - \ldots}} \quad (1)$$

with real coefficients $a_i$ converging to a limit $a$. Ramanujan claimed (see [1], p. 38) that if $a \neq \frac{1}{4}$, then the fraction converges if and only if $a < \frac{1}{4}$. The statement of convergence was proved in [6] for complex $a_i$ converging to $a \in \mathbb{C} \setminus [\frac{1}{4}, +\infty)$ (see also [4]). Gill [2] proved the divergence of (1) under the assumption that $a_i \to a > \frac{1}{4}$ fast enough, more precisely, whenever

$$\sum_i |a_i - a| < \infty. \quad (2)$$

The Ramanujan conjecture saying that (1) diverges always whenever $a_i \to a > \frac{1}{4}$ remained up to now an open question. In the present Note we disprove it. We show (Theorem 1.1) that for any $a > \frac{1}{4}$ there exists a real sequence $a_i \to a$ such that (1) converges.$^1$ Moreover, we show (Theorem 1.2) that Gill’s sufficient divergence condition (2) is the optimal condition on the speed of convergence of the $a_i$’s.

1.1. Main results

Theorem 1.1. For any $a > \frac{1}{4}$ there exists a real sequence $a_i \to a$ such that the continued fraction (1) converges.

The author did not find the Theorem in the literature. Its proof given below is elementary and it seems surprising that it was not discovered before. In the case, when $a_i \to \frac{1}{4}$, fraction (1) may also converge or diverge, see paper [3] and its bibliography.

Theorem 1.2. Let $q \in \mathbb{N}$, $q \geq 3$. Consider arbitrary sequence $r_i \to 0$, $r_i > 0$, such that $\sum_i r_i = \infty$. Then there exists an $a > \frac{1}{4}$ and a real sequence

$$a_i \to a, \quad a_i = a, \quad if \ i \not\equiv 1, 2 (\mod q), \quad a_{qi+1} - a = O(r_i), \quad a_{qi+2} - a = O(r_i), \quad as \ i \to \infty, \quad (3)$$

such that the continued fraction (1) converges.

$^1$ The author acknowledges that Alexey Tsygvintsev has constructed (by a completely different method) a beautiful explicit example [5] of a sequence $a_i \to 1$ corresponding to a convergent fraction (1) given by a simple recurrent formula. This example comes from the Analytic Function Theory.
1.2. The plan of the proofs and generalizations

In the proof of Theorem 1.1 the continued fraction (1) is expressed as a limit of compositions of Möbius transformations of the closed upper half-plane $\mathcal{H} = \{ \text{Im } z \geq 0 \}$ of the type

$$T_b : \mathcal{H} \to \mathcal{H}, \quad T_b(z) = -\frac{b}{z + 1}, \quad b \in \mathbb{R}.$$  

Proposition 1.3. The subsequent ratios $\frac{p_n}{q_n}$ of the continued fraction (1) are

$$\frac{p_n}{q_n} = \tau_n = T_{a_1} \circ \cdots \circ T_{a_n}(0).$$  

Proposition 1.3 is well-known and follows immediately from definition (by induction in $n$).

We recall the following

Definition 1.4. A Möbius transformation $M : \mathcal{H} \to \mathcal{H}$ of the upper half-plane is said to be elliptic (respectively, hyperbolic), if it has only one fixed point in $\text{Int}(\mathcal{H})$ (respectively, two fixed points on the boundary of $\mathcal{H}$, one of them is an attractor, the other one is a repeller). (An elliptic transformation is Möbius conjugated to a rotation of the unit disc. By definition, its rotation number is $(2\pi)^{-1}$ times the corresponding rotation angle.) The multiplier of a hyperbolic transformation $T$, denoted by $\mu = \mu(T)$, is its derivative at the attractor (by definition, $0 < \mu < 1$).

Remark 1. The transformation $T_a$ is elliptic if, and only if, $a > \frac{1}{4}$. Let $\rho(a)$ and $c(a)$ be the rotation number and the fixed point of $T_a$ in $\mathcal{H}$, respectively. The functions $\rho : (\frac{1}{4}, +\infty) \to (0, \frac{1}{2})$ and $c : (\frac{1}{4}, +\infty) \to i\mathbb{R} - \frac{1}{2}$ are analytic diffeomorphisms. One has

$$\rho(1) = \frac{1}{3},$$

since $T_1$ permutes cyclically 0, 1 and $\infty$. If $\rho(a) = \frac{p}{q} \in \mathbb{Q}$, then $T_a^q = \text{Id}$.

Example 1. Let $a_i \equiv a > \frac{1}{4}$. Then $\tau_n = T_a^n(0)$ does not have a limit: this sequence is either periodic or dense in $\mathbb{R}$.

Remark 2. By Gill’s result [2], if a Möbius transformation sequence $\phi_n : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ converges to an elliptic transformation, then the composition values $\tau_n = \phi_1 \circ \cdots \circ \phi_n(0)$ converge, provided that the modules of multipliers at fixed points of $\phi_n$ do not converge to 1 too fast and the fixed points converge fast enough. This does not apply to our situation of real $a_i \to a > \frac{1}{4}$, for which $\phi_i = T_{a_i}$ are elliptic.

Theorem 1.5. Theorem 1.1 holds for every $a$ with $\rho(a) \in \mathbb{Q}$. Let $\{T_a\}_{a \in \mathbb{R}} : \mathcal{H} \to \mathcal{H}$ be arbitrary smooth family of elliptic transformations (depending on a parameter $a$) with a smooth fixed point family $c(a)$. Let $a$ be a parameter value such that $\rho(a) = \frac{p}{q} \in \mathbb{Q} \setminus \{0, \frac{1}{2}\}$ and $c'(a) \neq 0$. Then there exists a sequence $a_i \to a$ such that the corresponding sequence $\tau_n$ from (4) converges. Moreover, for every divergent series $\sum r_i = \infty$, $r_i \to 0_+$, the sequence $a_i$ can be chosen to satisfy (3).

Theorem 1.6. Theorem 1.1 holds for each $a$ with $\rho(a) \notin \mathbb{Q}$. Moreover, let $T_a$ be arbitrary family of elliptic transformations as in Theorem 1.5. Let $a$ be a parameter value such that $\rho(a) \notin \mathbb{Q}$ and $\rho \neq \text{const } \text{near } a$. Then there exists a sequence $a_i \to a$ such that $\tau_n$ converge.

Theorems 1.5 and 1.6 imply Theorem 1.1. Theorem 1.5 implies Theorem 1.2. Theorem 1.5 is proved in the next section and Theorem 1.6 is proved in Section 3.

2. Limits with rational $\rho(a)$: proof of Theorem 1.5

Let $\rho(a) = \frac{p}{q} \in \mathbb{Q}$, thus, $T_a^q = 1$. We choose appropriate $\alpha_r$, $\beta_r \to a$ as specified below and...
set \( a_i = a \) if \( i \not\equiv 1, 2 \) (mod \( q \)); \( a_{qr+1} = \alpha_r, \ a_{qr+2} = \beta_r \). Denote
\[
T_{\alpha, \beta, q} = T_\alpha \circ T_\beta \circ T_a^{q-2}, \ \alpha, \beta \in \mathbb{R}. \quad (5)
\]
We choose \( \alpha_r, \beta_r \) so that \( T_{\alpha_r, \beta_r, q} \) are hyperbolic, let \( A_r, R_r \) be their attractors and repellers, respectively,
\[
A_i \to A, \ R_i \to R, \ A \neq R, \ R \notin C_a = \{ T_a^l(0), \ l = 0, \ldots, q - 1 \}, \quad (6)
\]
\[
\prod_r \mu_r = 0, \ \mu_r = \mu(T_{\alpha_r, \beta_r, q}). \quad (7)
\]
\[
\prod_r \mu_r = 0, \ \mu_r = \mu(T_{\alpha_r, \beta_r, q}). \quad (8)
\]
The existence of \( \alpha_r, \beta_r \) is proved at the end of this section.

We show in the next Proposition and the paragraph after that the corresponding sequence \( \tau_n \) converges, whenever conditions (6)–(8) hold.

**Proposition 2.1.** Let \( H_1, H_2, \ldots \) be an arbitrary sequence of hyperbolic transformations \( \mathcal{H} \to \mathcal{H} \) of the upper half-plane, \( A_i, R_i \) be respectively their attractors and repellers, \( A_i \to A, R_i \to R \neq A \). Let (8) hold with \( \mu_r = \mu(H_r) \). Then the mapping sequence \( \tilde{H}_n = H_1 \circ \cdots \circ H_n \) converges to a constant map uniformly on compact sets in \( \partial \mathcal{H} \setminus R \).

**Proof.** If \( A_1 = A, R_1 = R \), then the proposition follows immediately. If we fix a compact set \( K \subseteq \partial \mathcal{H} \setminus \{ A \cup R \} \), then for any \( i \) large enough the transformation \( H_i \) moves the points of \( K \) towards \( A \) (along \( \partial \mathcal{H} \)) by asymptotically the same distance, as the hyperbolic transformation with the same multiplier \( \mu_i \) but with \( A_i = A, R_i = R \). This together with the previous statement and the monotonicity of the restrictions \( H_i \mid_{\partial \mathcal{H}} \) implies the claim. \( \Box \)

### 2.1. Proof of convergence of \( \tau_n \)

If (6)–(8) hold, then the transformations \( H_r = T_{\alpha_r, \beta_r, q} \) satisfy the conditions of Proposition 2.1. Hence, their compositions \( \tilde{H}_r = T_{a_1} \circ T_{a_2} \circ \cdots \circ T_{a_q} \) converge uniformly to a constant limit (denote it \( x \)) on compact sets in \( \partial \mathcal{H} \). By definition, \( \tau_{qr} = \tilde{H}_r(0), 0 \neq R \) by (7). This implies that \( \tau_{qr} \to x \), as \( r \to \infty \). To show that the whole sequence \( \tau_n \) converges to \( x \), we use condition (7), which says that the finite \( T_{a_q} \)-orbit \( C_a \) of 0 does not meet \( R \). Let \( \delta > 0 \) be such that the closed \( \delta \)-neighborhood \( U \) of the latter orbit be disjoint from \( R \). Then \( \tilde{H}_r \to x \) uniformly on \( U \). If \( r \) is large enough, then
\[
T_{\alpha_r}(0), T_{\alpha_r} \circ T_{\beta_r}(0), T_{\alpha_r} \circ T_{\beta_r} \circ T_a^{l-2}(0) \in U \ (0 < l \leq q).
\]
By definition, \( \tau_{qr+l} = \tilde{H}_{r+l} \circ \tilde{T}_{\alpha_l} \circ \tilde{T}_{\beta_l} \circ T_a^{l-2}(0) \). The two last statements imply together that the \( q \) sequences \( \tau_{qr}, \tau_{qr+1}, \tau_{qr+2}, \ldots, \tau_{qr+q-1} \) converge to \( x \), hence, the whole sequence \( \tau_n \) converges. Below we show that for arbitrary given \( r_i > 0, \sum r_i = \infty \), one can achieve that in addition to (6)–(8), condition (3) holds. This will prove Theorems 1.5 and 1.2.

### 2.2. The construction of sequences \( \alpha_i, \beta_i \) satisfying (6)–(8)

**Lemma 2.2** (Main Technical Lemma). Let a family \( T_a \) and a parameter value \( a \) be as in Theorem 1.5. Then for any point \( R \in \partial \mathcal{H} \) (may be except two points) there exist two linear families of parameter values
\[
\alpha(t) = a + c_1 t, \ \beta(t) = a + c_2 t, \ c_1, c_2 \in \mathbb{R}, \quad (9)
\]
such that for any \( t > 0 \) small enough the transformation \( T_{\alpha(t), \beta(t), q} = T_{\alpha(t)} \circ T_{\beta(t)} \circ T_a^{q-2} \) be hyperbolic and its repeller \( R(t) \) (respectively, attractor \( A(t) \)) tends to \( R \) (respectively, to a point \( A \neq R \)), as \( t \to 0_+ \). Moreover, one can achieve that the families \( A(t), R(t) \) be smooth at 0, and the derivative in \( t \) at \( t = 0 \) of the previous family \( T_{\alpha(t), \beta(t), q} \) be nonzero.

The lemma is proved below.

Let \( \rho(a) = \frac{p}{q}, C_a \) be the (finite) \( T_a \)-orbit of 0. Let us choose any \( R \notin C_a \) that satisfies the statements of Lemma 2.2. Let \( \alpha(t), \beta(t) \) be the corresponding families (9). Take a sequence \( t_k \to 0_+ \) and put \( \alpha_k = \alpha(t_k), \beta_k = \beta(t_k) \). The conditions (6) and (7) follow immediately from construction. Condition (8) holds, if and only if \( \sum t_i = \infty \) (these...
are the $t_i$ we choose). This follows from the fact that the function $\mu(t) = \mu(T_{a(t),\beta(t),q})$ has nonzero derivative at $0$: $\mu(t) = 1 + st + O(t^2)$, $s \neq 0$, hence, $\ln \mu_k = sk(1 + o(1))$. Indeed, otherwise, if $\mu'(0) = 0$, the transformation family $T_{a(t),\beta(t),q}$ would have zero derivative in $t$ at $t = 0$ – a contradiction to the hypotheses of the lemma. This finishes the construction. Statement (3) follows immediately, if we put $t_i = r_i$. Theorem 1.5 is proved.

**Proof of Lemma 2.2.** Consider $T_{a,\beta,q} = T_\alpha \circ T_\beta \circ T_{a}^{-q}$ as a family of mappings depending on two parameters $\alpha$ and $\beta$ (the $a$ is fixed). It is identity, if $\alpha, \beta = a$. Consider its derivatitives in $\alpha$ ($\beta$) at $\langle \alpha, \beta \rangle = (a, a)$ as vector fields on $\partial H$ denoted $v_1$ (respectively, $v_2$). It follows from definition that $v_2 = (T_a)_{*}v_1$. We claim that the fields $v_1$ and $v_2$ are not constant-proportional. Hence, for any point $R \in \partial H$ one can find a linear combination $v = c_1v_1 + c_2v_2 \neq 0$ that vanishes at $R$. If the 1-jet of $v$ at $R$ does not vanish (then one can achieve that $v'(R) > 0$ by changing sign), this implies that $v$ has another zero $A \in \partial H \setminus R$. Then the corresponding families (9) are those we are looking for. If the latter 1-jet vanishes, this implies that the commutator $[v_1, v_2]$ (which also belongs to the Lie algebra of the group $\text{Aut}(H)$) vanishes at $R$. The latter commutator does not vanish identically (since $v_1$ are not proportional) and cannot have more than two zeros. This together with the previous discussion proves the lemma. \hfill $\square$

3. **Case of irrational limit rotation**

**Proof of Theorem 1.6.** Let $\rho(a) \notin \mathbb{Q}$, $\tilde{a}_n \to a$, $\rho(\tilde{a}_n) = \frac{\tilde{a}_n}{q_n} \in \mathbb{Q}$. We choose appropriate $\alpha_n$, $\beta_n \to a$, a natural number sequence $N_1, N_2, \ldots$ and define $\tilde{a}_n$ as follows:

1) For $n \leq N_1 q_1$ set $a_n = \tilde{a}_1$, if $n \neq 1, 2 \pmod{q_1}$; $a_{q_1 r + 1} = \alpha_1$, $a_{q_1 r + 2} = \beta_1$, $r < N_1$.

2) Let $N_1 q_1 < n \leq N_1 q_1 + N_2 q_2$. Put $n_1 = n - N_1 q_1$, $a_n = \tilde{a}_2$, if $n_1 \neq 1, 2 \pmod{q_2}$; $a_n = \alpha_2$, if $n_1 \equiv 1 \pmod{q_2}$; $a_n = \beta_2$, if $n_1 \equiv 2 \pmod{q_2}$, etc.

We show that (1) converges if we take $\alpha_i$, $\beta_i$ and $N_i$ as specified below.

**Choice of $\alpha_n$ and $\beta_n$.** Denote $\psi_{0} = \psi_{n,0} = \text{Id}$, $\psi_{n} = T_{a_n,\beta_n,q_n} = T_{a_n} \circ T_{\beta_n} \circ T_{a}^{-q_n}$. $\psi_{n,1} = T_{\alpha_n}, \psi_{n,2} = T_{\beta_n}$, $\psi_{n,l} = \psi_{n,2} \circ T_{a}^{-l}$, for $2 \leq l \leq q_n - 1$. We choose $\alpha_n$ and $\beta_n$ so that the transformations $\psi_{R_n}$ be hyperbolic, denote $A_n$, $R_n$ their attractors and repellers, respectively;

$$R_n \notin M_n = \{A_{n,1}, \psi_{n,1}(A_{n,1}), 0, \psi_{n,1} \circ \psi_{n,1}(0) \mid 0 \leq l \leq q_n - 1, r \in \mathbb{N} \cup \{0\} \}.$$ (11)

**Remark 3.** The set $M_n$ in (11) is infinite and accumulates exactly to the finite $T_{a_{n}}$-orbit of $A_{n+1}$, which follows from definition. This implies that if (11) holds, then $M_n$ does not accumulate to $R_n$. Thus, in this case choosing appropriate power $N_n$, one can achieve that the image $\psi_{n}^{N_n}(M_n)$ be arbitrarily close to $A_n$.

**Choice of $N_i$.** The parameters $\alpha_i$, $\beta_i$ satisfy (10), (11). Put $\theta_k = \psi_{1}^{N_1} \circ \cdots \circ \psi_{k}^{N_k}$. We construct $N_i$ (by induction in $i$) in such a way that

$$\text{diam}(\theta_k(M_k)) < \frac{1}{2^k}. \quad \text{(12)}$$

It is possible by the last statement of the previous remark. Let us show that then the sequence $\tau_n$ is Cauchy (hence, converges). Denote $n_k = \sum_{i=1}^{k} q_i N_i$. It suffices to show that for any $k$ and any $m \geq n_k$ one has $\text{dist}(\tau_{n_k}, \tau_m) < \frac{1}{2^{k+2}}$. \hfill (13)

Case 1: $m = n_i > n_k$, say, $m = n_{k+1}$. Then $\tau_{n_k} = \theta_k(0)$, $\tau_m = \theta_{k+1}(0) = \theta_k \circ \psi_{k+1}^{N_{k+1}}(0)$. By definition, $0, \psi_{k+1}^{N_{k+1}}(0) \in M_k$. By (12), $\text{dist}(\theta_k(0), \theta_{k+1}(0)) = \text{dist}(\theta_k(0), \theta_k(\psi_{k+1}^{N_{k+1}}(0))) < \frac{1}{2^k}$. Therefore,
\[ \text{dist}(\theta_k(0), \theta_s(0)) < \frac{1}{2^{k-1}} \quad \text{for any } s > k. \quad (14) \]

This proves (13) for any \( m = n_i > n_k \).

General case: \( m > n_k \) is arbitrary. Take \( s \in \mathbb{N} \) such that \( n_s \leq m < n_{s+1} \). Then \( m = n_s + rq_{s+1} + l, \quad 0 \leq l < q_{s+1}, \quad \tau_m = \theta_s \circ \psi_{s+1}^{-1} \circ \psi_{s+1,l}(0) \). Analogously to the previous discussion, by (12), \( \text{dist}(\theta_s(0), \tau_m) < \frac{1}{2^s} \). This together with (14) implies (13). Theorem 1.6 is proved. \( \square \)

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