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Algebra

The fundamental group of a triangular algebra without double bypasses

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Abstract

Let A be a basic connected finite dimensional algebra over a field of characteristic zero. A fundamental group depending on the presentation of A has been defined by several authors [see R. Martínez-Villa, J.A. de La Peña, The universal cover of a quiver with relations, J. Pure Appl. Algebra 30 (1983) 277–292]. Assuming the quiver of A has no oriented cycles and no double bypasses, we show there exists a suitable presentation of A with quiver and admissible relations, with fundamental group denoted by $\pi_1(A)$, such that the fundamental group of any other presentation of A with quiver and admissible relations is a quotient of $\pi_1(A)$. ***To cite this article: P. Le Meur, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Le groupe fondamental d'une algèbre triangulaire sans double raccourci. Soit A une algèbre basique connexe et de dimension finie sur un corps de caractéristique nulle. Plusieurs auteurs [voir R. Martínez-Villa, J.A. de La Peña, The universal cover of a quiver with relations, J. Pure Appl. Algebra 30 (1983) 277–292] ont défini pour A un groupe fondamental dépendant du choix d'une présentation de A. En supposant que le carquois de A n'a pas de cycle orienté et n'a pas de double raccourci, nous démontrons qu'il existe une présentation privilégiée de A par carquois et relations admissibles, de groupe fondamental noté $\pi_1(A)$, telle que le groupe fondamental de toute autre présentation de A par carquois et relations admissibles est un quotient de $\pi_1(A)$. ***Pour citer cet article : P. Le Meur, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

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Soit A une algèbre basique, connexe et de dimension finie sur un corps k, et soit Q le carquois (connexe) de A. Si I est un idéal admissible de kQ tel que $A \simeq kQ/I$, le groupe fondamental $\pi_1(Q,I)$ de (Q,I) a été défini dans [7] à partir de la relation d'homotopie $\sim_I$ de (Q,I). Il existe des exemples d'algèbres A avec différentes présentations admissibles $A \simeq kQ/I$ et $A \simeq kQ/J$ telles que $\pi_1(Q,I) \not\simeq \pi_1(Q,J)$. Le but de ce texte est d'étudier cette situation et la possible existence d'un groupe fondamental qui serait canoniquement attaché à cette algèbre. Pour cela nous considérons les raccourcis de Q c'est-à-dire les couples (α,u) où α est une flèche de Q et u est un chemin (orienté) de Q parallèle à α et distinct de α. Un double raccourci de Q est un quadruplet (α,u,β,v) où (α,u) et (β,v) sont deux raccourcis tels que β est une flèche parcourue par le chemin u. Dans cette note, nous montrons le théorème suivant :

Théorème 0.1. *Supposons que le corps k est de caractéristique nulle. Supposons que Q n'a pas de cycle orienté et que Q n'admet pas de double raccourci. Alors il existe une présentation admissible $kQ/I_0 \simeq A$ de A telle que pour toute autre présentation admissible $kQ/I \simeq A$ il existe un morphisme surjectif de groupes $\pi_1(Q,I_0) \twoheadrightarrow \pi_1(Q,I)$.*

Notons que l'hypothèse sur les doubles raccourcis implique que le carquois Q n'admet pas de flèches parallèles. Si $kQ/I \simeq A$ et $kQ/J \simeq A$ sont deux présentations admissibles de A telles que $\sim_I$ est plus fine que $\sim_J$ (i.e. $\gamma \sim_I \gamma' \Rightarrow \gamma \sim_J \gamma'$), alors il existe un morphisme surjectif de groupes $\pi_1(Q,I) \twoheadrightarrow \pi_1(Q,J)$. Partant de cette remarque, la preuve du théorème consiste à construire un carquois Γ dont les sommets sont les relations d'homotopie des présentations admissibles de A et tel que si $\sim_I \to \sim_J$ est une flèche de Γ, alors $\sim_I$ est strictement plus fine que $\sim_J$. Avant de décrire les flèches de Γ, nous introduisons des automorphismes particuliers de kQ : les transvections et les dilatations. Une dilatation est un automorphisme $\varphi : kQ \to \sim kQ$ tel que $\varphi(\alpha) \in k^*\alpha$ pour toute flèche α. Une transvection est un automorphisme de la forme $\varphi_{\alpha,u,\tau} : kQ \to \sim kQ$ où (α,u) est un raccourci, $\tau \in k$ et $\varphi_{\alpha,u,\tau}$ est défini par $\varphi_{\alpha,u,\tau}(\alpha) = \alpha + \tau u$ et $\varphi_{\alpha,u,\tau}(\beta) = \beta$ pour toute flèche $\beta \neq \alpha$. L'intérêt des dilatations et des transvections est le suivant : soit $kQ/I \simeq A$ une présentation admissible de A, soit $\varphi : kQ \to kQ$ un automorphisme et soit $J = \varphi(I)$. Si φ est une dilatation alors $\sim_I$ et $\sim_J$ coïncident, si φ est une transvection alors l'une des deux relations d'homotopie $\sim_I$ ou $\sim_J$ est plus fine (au sens large) que l'autre. Cette propriété permet de définir les flèches de Γ : il existe une flèche $\sim \to \sim'$ dans Γ si et seulement si il existe deux présentations admissibles $kQ/I \simeq A$ et $kQ/J \simeq A$ ainsi qu'une transvection $\varphi : kQ \to kQ$ telles que $\sim = \sim_I$, $\sim' = \sim_J$, $J = \varphi(I)$ et $\sim_I$ est strictement plus fine que $\sim_J$. De cette façon, Γ est un carquois connexe, la longueur des chemins orientés de Γ est bornée et tout sommet de Γ est le but d'un chemin orienté dont la source est une source de Γ (i.e. n'est le but d'aucune flèche de Γ). Nous montrons alors que le carquois Γ n'a qu'une seule source et que si $kQ/I_0 \simeq A$ est une présentation admissible de A telle que $\sim_{I_0}$ est l'unique source de Γ, alors le couple (Q,I_0) satisfait la conclusion du théorème. Il est à noter que si $\mathrm{car}(k) \neq 0$, il existe des exemples d'algèbre A dont le carquois Q n'a ni cycle orienté ni double raccourci et telle que le carquois Γ admet plusieurs sources (le théorème reste cependant vrai pour ces exemples). La preuve de ce théorème sera détaillée dans un article en préparation qui fera partie de la thèse de l'auteur à Montpellier sous la direction de Claude Cibils, le cadre de travail sera alors élargi au contexte des revêtements galoisiens de l'algèbre A.

1. Introduction

Let A be a finite dimensional connected algebra over a field k. We are interested in the representation theory of A. Thus we may assume A is basic and we denote by Q its ordinary quiver. Any presentation $kQ/I \simeq A$ of A with quiver and admissible relations (i.e. I is an admissible ideal of kQ and $kQ/I \simeq A$ is an isomorphism of k-algebras) gives rise to a group $(\pi_1(Q,I))$ called the fundamental group of (Q,I) (see [3,7]). The fundamental group is particularly useful in covering techniques (see [3–5]). However different presentations of A with quiver and admissible relations may provide non isomorphic fundamental groups. In this text we intend to clarify this *uncanonical* situation. For this purpose we will use the notion of double bypass. Recall that a bypass (see [1]) is a pair (α, u) where α is an arrow of Q and u is an oriented path of Q parallel to α and distinct from α. A double bypass is quadruple (α, u, β, v) where (α, u) and (β, v) are bypasses such that the arrow β appears in the path u. Assume that $\mathrm{car}(k) = 0$ and suppose the algebra A satisfies:

- A is triangular (i.e. Q has no oriented cycles),
- the quiver Q has no double bypasses.

Theorem 1.1. *Assuming the above conditions, there exists a presentation $kQ/I_0 \simeq A$ with quiver and relations such that for any other presentation $kQ/J \simeq A$ there is a surjective group morphism $\pi_1(Q, I_0) \to \pi_1(Q, J)$.*

Notice that our assumption on double bypasses implies that Q has no double arrows. Indeed, if $\alpha \neq \beta$ are parallel arrows, then $(\alpha, \beta, \beta, \alpha)$ is a double bypass. Notice also that if we assume moreover that A is Schurian, we recover the result [2, Theorem 3.5].

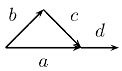
This Note is part of the author's thesis made at Université Montpellier 2 under the supervision of Claude Cibils. A detailed version of this note will be written in a subsequent paper and in the author's thesis report.

2. Basic definitions and notations

Let (Q, I) be a connected *quiver with admissible relations*, i.e. Q is a (finite) connected quiver Q and I is an admissible ideal of kQ. Recall that admissible means $(kQ^+)^n \subseteq I \subseteq (kQ^+)^2$ for some n (where $(kQ^+)^n$ stands for the ideal generated by paths of length n). A *walk* is an unoriented path of Q. The stationary walk at a vertex x will be denoted by e_x. Let $r = t_1u_1 + \cdots + t_nu_n \in I$ where $t_i \in k^*$ and the u_i's are distinct paths. Then r is called a *minimal relation* if $n \geqslant 2$ and if for any non empty proper subset E of $\{1, \ldots, n\}$ the term $\sum_{i \in E} t_i \cdot u_i$ does not lie in I. The *homotopy relation* $\sim_I$ of (Q, I) is the smallest equivalence relation on the set of walks (of Q) which is compatible with the concatenation of walks and such that:

(i) for any arrow α with source x and target y, we have $\alpha\alpha^{-1} \sim_I e_y$ and $\alpha^{-1}\alpha \sim_I e_x$;
(ii) $u_1 \sim_I u_2$ as soon as $t_1u_1 + \cdots + t_nu_n$ is a minimal relation of (Q, I).

Let then x_0 be a vertex of Q. The *fundamental group* (see [3,7]) $\pi_1(Q, I, x_0)$ of (Q, I) at x_0 is the set of $\sim_I$-homotopy classes of walks starting and ending at x_0. The composition in $\pi_1(Q, I, x_0)$ is induced by the concatenation of walks and the unit is the $\sim_I$-class of e_{x_0}. As the choice of x_0 is irrelevant (since Q is connected) we will write $\pi_1(Q, I)$ for short.

Example 1. Assume Q is equal to 

and set $I = \langle da \rangle$ and $J = \langle da - dcb \rangle$.

Then $kQ/I \simeq kQ/J$ whereas $\pi_1(Q, I) \simeq \mathbb{Z}$ and $\pi_1(Q, J) = 0$.

3. Preliminary results

As we wish to compare the fundamental group of different presentations of a given algebra, it is natural to try to compare the corresponding homotopy relations. In this section we give two helpful lemmas for such a comparison. Let us first introduce some terminology. Let Q be any quiver with set of vertices Q_0 and set of arrows Q_1. Let $\varphi : kQ \to kQ$ be an automorphism which is the identity map on Q_0. We will say that φ is a *dilatation* if $\varphi(\alpha) \in k^*\alpha$

for any $\alpha \in Q_1$. We will say that φ is a *transvection* if there exists a bypass (α, u) and $\tau \in k$ such that $\varphi = \varphi_{\alpha,u,\tau}$ where $\varphi_{\alpha,u,\tau}(\alpha) = \alpha + \tau u$ and $\varphi_{\alpha,u,\tau}(\beta) = \beta$ for $\beta \in Q_1 \backslash \{\alpha\}$. In analogy with the classical decomposition of elements of $\mathrm{GL}_n(k)$ as products of transvections and dilatations, we have the following lemma.

Lemma 3.1. *Assume Q has no oriented cycles and let (Q, I) and (Q, J) be two quivers with admissible relations satisfying $kQ/I \simeq kQ/J$. Then there exists $\psi : kQ/I \to kQ/J$ an isomorphism which is the identity map on Q_0. Moreover for any such ψ, there exists $\varphi : kQ \to kQ$ an automorphism such that $\varphi(I) = J$ and such that φ induces ψ. Finally φ is a composition of a dilatation and finitely many transvections.*

Note that Lemma 3.1 can (partially) be rewritten in terms of semi-direct products of groups as follows: the group of automorphisms $kQ \to kQ$ which are the identity map on Q_0 is the semi-direct product of the subgroup of dilatations with the (normal) subgroup generated by transvections. Notice also that other studies of the automorphism group of an algebra were made in relation with the Picard group of the algebra (see [6,8–10]).

We now turn to a fundamental lemma which is the first important step in the proof of Theorem 1.1.

Lemma 3.2. *Assume Q has no oriented cycles and let (Q, I) and (Q, J) be quivers with admissible relations.*

Let $\varphi : kQ \to kQ$ be an automorphism with $\varphi(I) = J$. If φ is a dilatation then $\sim_I$ and $\sim_J$ coincide. Assume now $\varphi = \varphi_{\alpha,u,\tau}$ is a transvection.

(a) *If $\alpha \sim_I u$ and $\alpha \sim_J u$ then $\sim_I$ and $\sim_J$ coincide.*
(b) *If $\alpha \not\sim_I u$ and $\alpha \sim_J u$ then $\sim_J$ is generated by $\sim_I$ and $\alpha \sim_J u$.*
(c) *If $\alpha \not\sim_I u$ and $\alpha \not\sim_J u$ then $\sim_I$ and $\sim_J$ coincide and $I = J$.*

Remark 1. The word *generated* stands for: generated as an equivalence relation which is compatible with the concatenation of walks and such that $\alpha\alpha^{-1} \sim e_y$ and $\alpha^{-1}\alpha \sim e_x$ for any arrow $x \xrightarrow{\alpha} y \in Q_1$.

Remark 2. The following implication (symmetrical to (b)):

If $\alpha \not\sim_J u$ and $\alpha \sim_I u$ then $\sim_I$ is generated by $\sim_J$ and $\alpha \sim_I u$

is also true: apply point (b) after exchanging I and J and after replacing φ by $\varphi^{-1} = \varphi_{\alpha,w,-\tau}$.

If $\sim$ and $\sim'$ are homotopy relations, we will say $\sim'$ is a *direct successor* of $\sim$ if there exist quivers with admissible relations (Q, I) and (Q, J) presenting A together with $\varphi_{\alpha,u,\tau}$ a transvection such that: $\sim_I = \sim$, $\sim' = \sim_J$, $J = \varphi_{\alpha,u,\tau}(I)$, $\alpha \not\sim_I u$ and $\sim_J = \langle \sim_I, u \sim_J \alpha \rangle$. Notice that there may exist various (Q, I), (Q, J) and $\varphi_{\alpha,u,\tau}$ providing the same homotopy relations and a direct successor relation between them. These remarks will be taken into account in the following definition.

4. Proof of Theorem 1.1

Assume A is a finite dimensional basic and connected k-algebra with ordinary quiver Q.

Definition 4.1. If Q has no oriented cycles, we define a quiver Γ as follows:

- the vertices of Γ are the homotopy relations of the admissible presentations of A with quiver and relations,
- Γ has an arrow $\sim \to \sim'$ if and only if $\sim'$ is a direct successor of $\sim$.

The author thanks Mariano Suárez-Alvarez for the following remark:

Remark 3. A homotopy relation is determined by its restriction to the paths of Q with length at most the radical length of A, thus there are only finitely many homotopy relations. This argument shows that Γ is finite. As a consequence, any vertex of Γ is the target of a (finite) oriented path in Γ with source a source of Γ (i.e. a vertex with no arrow arriving at it).

Moreover, using Lemma 3.2, the following result gives additional properties of the quiver Γ.

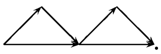
Proposition 4.2. *Assume Q has no oriented cycles and let m be the number of bypasses in Q. Then Γ is connected and has no oriented cycles. Any vertex of Γ is the source of at most m arrows. The length of the oriented paths in Γ is bounded by m.*

Suppose now that $\sim_{I_0} \to \cdots \to \sim_{I_n}$ is an oriented path in Γ. For each i, the natural map $\pi_1(Q, I_i) \to \pi_1(Q, I_{i+1})$ induced by the identity map on walks is a well defined surjective group morphism. Thus there is a surjective group morphism $\pi_1(Q, I_0) \twoheadrightarrow \pi_1(Q, I_n)$. Consequently, it is natural to ask whether Γ has a unique source. In case of a positive answer, this unique source gives rise to a canonical homotopy relation among all other homotopy relations of the presentations of the given algebra A. The following proposition answers this question. Notice that the preceding results did not use the hypotheses (written before stating Theorem 1.1) concerning car(k) or concerning the double bypasses.

Proposition 4.3. *Assume A satisfies the hypotheses written before stating Theorem* 1.1*, then Γ has a unique source.*

The proof of Theorem 1.1 is now straightforward: let $kQ/I_0 \simeq A$ be a presentation of A such that $\sim_{I_0}$ is the unique source of Γ. For any other homotopy relation $\sim_J$ (with $kQ/J \simeq A$), there exists a path $\sim_{I_0} \to \cdots \to \sim_J$ in Γ, thus we have a surjective group morphism $\pi_1(Q, I_0) \twoheadrightarrow \pi_1(Q, J)$.

Remark 4. If m is the number of bypasses of Q, the unicity of the source of Γ implies that Γ has at most $1 + m + m^2 + \cdots + m^m$ vertices. In particular, under the assumptions made before stating Theorem 1.1, there are at most $1 + m + m^2 + \cdots + m^m$ isomorphism classes of groups which can be the fundamental group of a presentation of A with quiver and admissible relations.

Remark 5. Let $Q =$ . Notice that Q has no double bypasses. Let u (resp. v) be the path parallel to a (resp. α), let $I_1 = \langle \alpha a + vu, va + \alpha u \rangle$ and set $A \simeq kQ/I_1$. Then $\pi_1(Q, I_1) = \mathbb{Z}/2$. Let $I_2 = \varphi_{a,u,-1}\varphi_{\alpha,v,-1}(I_1)$, hence $A \simeq kQ/I_2$. If car(k) $= 0$, then $I_2 = \langle \alpha a, va + \alpha u - 2vu \rangle$, $\pi_1(Q, I_2) = 0$ and $\sim_{I_1}$ is the unique source of Γ. Suppose now that car(k) $= 2$. Then $I_2 = \langle \alpha a, va + \alpha u \rangle$ and $\sim_{I_1}$ and $\sim_{I_2}$ are both sources of Γ whereas $\sim_{I_1} = \langle \sim_{I_2}, \alpha a \sim_{I_1} vu \rangle$ does not coincide with $\sim_{I_2}$. Notice that we still have a surjection $\pi_1(Q, I_2) = \mathbb{Z} \twoheadrightarrow \mathbb{Z}/2 = \pi_1(Q, I_1)$. Notice also that one can build similar examples for any nonzero value p of car(k) by taking for Q a sequence of p bypasses instead of 2 bypasses only.

Remark 6. The details of our proof of Theorem 1.1 show that the assumption on car(k) can be weakened. More precisely, if $p =$ car(k) $\neq 0$ and if the quiver Q has less than p bypasses, then Theorem 1.1 still holds.

The results presented here can be reformulated into results on Galois coverings (where the group $\pi_1(A)$ corresponds to the universal cover of A and which does not depend on the presentation of A). This reformulation will be made in a subsequent paper.

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