Probability Theory

Riemannian connections and curvatures on the universal Teichmüller space

Hélène Airault\textsuperscript{a,b}

\textsuperscript{a} INSET, université de Picardie, 48, rue Raspail, 02100 Saint-Quentin, France
\textsuperscript{b} Laboratoire CNRS UMR 6140 LAMFA, 33, rue Saint-Leu, 80039 Amiens, France

Received 3 June 2005; accepted 20 June 2005
Available online 3 August 2005
Presented by Paul Malliavin

Abstract

We define Riemannian connections on the universal Teichmüller space $U^\infty$. For the Levi-Civita’s connection on $U^\infty$, the Riemannian curvature tensor is well defined and the Ricci curvature is finite. We obtain several series of infinite dimensional operators which converge. To cite this article: H. Airault, C. R. Acad. Sci. Paris, Ser. I 341 (2005).

Résumé


Version française abrégée

Soit $\text{Diff}(S^1)$ le groupe des difféomorphismes du cercle qui pré servent l’orientation et soit $\mathcal{H}$ le sous-groupe des transformations homographiques. Un difféomorphisme $e^{\gamma \theta}$ s’identifie avec l’application $\gamma : \theta \to \gamma(\theta)$ modulo $2\pi$. Soit $\text{diff}(S^1)$ l’algèbre de Lie de $\text{Diff}(S^1)$. L’algèbre de Lie de $\mathcal{H}$ est notée $\text{su}(1, 1)$, elle est engendrée par $\cos \theta$, $\sin \theta$, 1. Pour $k$ un entier, $k \geq 0$, on pose $\alpha(k) = ak^3 + bk$ où $a \geq 0$ et $b$ est un nombre réel. Sur l’espace vectoriel $V$ des séries de Fourier $u(\theta) = \sum_{k \geq 0} a_k \cos(k\theta) + b_k \sin(k\theta)$ telles que $\|u\|^2 = \sum_{k \geq 1} (a_k^u)^2 \alpha(k) + (b_k^u)^2 \alpha(k) < +\infty$, on considère le produit scalaire

E-mail address: helene.airault@insset.u-picardie.fr (H. Airault).

1631-073X/$ – see front matter © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.
séries formées par des opérateurs diagonaux dans la base orthonormale
La courbure de cette connexion est égale à celle de la connexion obtenue dans [6]. On étudie la convergence de
$C^2$.

(1)

Diffe $(S^U)$ we introduce different Riemannian connections on $S^U$. Among them the Levi-Civita connection (4) which
commutes with the Hilbert transform $J$.

This leads to the Riemannian curvature and Ricci curvature of $U^\infty$.

\[
(u|v) = \sum_{k \geq 1} a_k^u b_k^v \alpha(k) + b_k^u a_k^v \alpha(k).
\]

Comme $(\cdot|\cdot) = \sum_{k \geq 2} (a_k^u b_k^v - a_k^v b_k^u) k \alpha(k), on a (w'|u) = -(w|u')$. Pour $u, v, w \in V,$
\[
[u, v](\theta) = u(\theta)v'(\theta) - u'(\theta)v(\theta).
\]

Lorsque $\alpha(k) = k^3 - k$ et pour $u, v, w$ dans le sous-espace vectoriel $V_0$ engendré par $\{\cos(k\theta); \sin(k\theta)\}_{k \geq 2}$, on démontre l’identité remarquable
\[
(Ju | [v, w]) + (Jv | [u, w]) + (Jw | [u, v]) = 0,
\]

où $J$ est la transformation de Hilbert, $J \cos k\theta = \sin k\theta$ et $J \sin k\theta = -\cos k\theta, k \geq 1$. Soit $\pi$ la projection de
diff$(S^1)$ sur $V_0$. Lorsque $u, v, w \in$ diff$(S^1)$, on définit la connexion de Levi-Civita sur le groupe quotient $\mathcal{H}/\text{Diff}(S^1)$ par (cf. [12, vol. 2, p. 201])
\[
\Gamma_{\mathcal{L}C}(u, v, w) = (\Gamma_{\mathcal{L}C}(v)u | w) = \frac{1}{2} [([u, v] | \pi v) + ([w, v] | \pi u) - ([u, v] | \pi w)].
\]

La courbure de cette connexion est égale à celle de la connexion obtenue dans [6]. On étudie la convergence de
séries formées par des opérateurs diagonaux dans la base orthonormale $[c_k = \cos(k\theta); s_k = \sin(k\theta)]_{k \geq 2}$ et issues de
différentes connexions. Soit $\Gamma_{\gamma}(v, u)[w, v]$. La série d’opérateurs $\sum_{j \geq 2} \Gamma_{\mathcal{L}C}(e_j)^2 + \Gamma_{\mathcal{L}C}(s_j)^2$ converge et la série
\[
A = \sum_{j \geq 2} (\Gamma_{\mathcal{L}C} + \Gamma_1)(e_j) \Gamma_{\mathcal{L}C}(e_j) + (\Gamma_{\mathcal{L}C} + \Gamma_1)(s_j) \Gamma_{\mathcal{L}C}(s_j)
\]
définit un opérateur de courbure borné.

1. Introduction

Among the difficulites of geometry in infinite dimension are the importance of a good choice of coordinate system
and the divergence of series associated to the operation of contraction. Infinite-dimensional geometry appears
naturally in Probability Theory [1,2,7] and in Mathematical Physics [6,5,11]. The universal Teichmüller space $U^\infty$ is
the space of $C^\infty$ Jordan curves of the complex plane. In our program, $U^\infty$ will appear as the skeleton of canonical
probability measures which will be carried by a larger space than $U^\infty$ (see [4,10]). To $\gamma \in U^\infty$, correspond
two univalent functions $f^+_{\gamma}, f^-_{\gamma}$ sending the inside and the outside of the unit disk on the two regions delimited
by $\gamma$; then $f^+_{\gamma}, f^-_{\gamma}$ are $C^\infty$ functions on the circle $S^1$. Therefore, $f_{\gamma}(\theta) := ([f^+_{\gamma}]^{-1} \circ f^-_{\gamma}) \exp(i\theta)$ is a diffeomorphism of the circle. Denote Diff$(S^1)$ the group of $C^\infty$, orientation preserving diffeomorphisms of the circle. Let $\mathcal{H}$ be the subgroup of Möbius transformations of the disk considered as operating on $S^1$; then $f^+_{\gamma}, f^-_{\gamma}$ are defined
up to a Möbius transformation. The previous construction defines an injective map $U^\infty \rightarrow \mathcal{H}/\text{Diff}(S^1)/\text{Rot}(S^1)$.
The Beurling–Ahlfors theory of conformal welding shows that this map is surjective. The Lie algebra diff$(S^1)$ is the
set of $C^\infty$ vector fields on the circle, they are identified with the $C^\infty$ functions on the circle. The Lie algebra su(1, 1) of $\mathcal{H}$ is the linear subspace of diff$(S^1)$ having for basis 1, $\cos \theta, \sin \theta$. It has been shown in [3] that there exists a unique scalar product on diff$(S^1)$ which is invariant under the adjoint action of su(1, 1). The associated
metric defines a canonical Riemannian structure on $U^\infty$. An orthonormal basis for this metric at identity is
\[
\left\{ \frac{1}{\sqrt{k^3-k}} \cos k\theta, \frac{1}{\sqrt{k^3-k}} \sin k\theta \right\}, k > 1.
\]

(i) we prove the identity (3) which is specific to the metric $\alpha(k) = k^3 - k$,
(ii) we introduce different Riemannian connections on $U^\infty$, among them the Levi-Civita connection (4) which
commutes with the Hilbert transform $J$.\[\]
2. The geometry of $\mathcal{H}\setminus\text{Diff}(S^1)$

We calculate on $\text{Diff}(S^1)$ modulo composition on the left by homographic transformations. Let $\gamma \in \text{Diff}(S^1)$ and $u(\theta) \in V$. To obtain right invariant vector fields, we define for small $\epsilon > 0$, $\gamma_\epsilon(\theta) = \gamma(\theta) + \epsilon u(\gamma(\theta)) = (\exp(\epsilon u) \circ \gamma)(\theta) + o(\epsilon^2)$, then $X_u(\gamma) = \frac{d}{d\epsilon}|_{\epsilon=0}\gamma_\epsilon = u \circ \gamma$ is right invariant since for the right translation $R(\gamma) = \gamma \circ \gamma_1$, we have $dR(\gamma)[X_u(\gamma)] = X_u(\gamma \circ \gamma_1) = X_u(\gamma) \circ \gamma_1$. In the same way, we put $\gamma_\epsilon(\theta) = \gamma(\theta + \epsilon u(\theta)) = (\gamma \circ \exp(\epsilon u))(\theta) + o(\epsilon^2)$, then $Y_u(\gamma) = \epsilon u(\theta)$ where $\gamma_\epsilon'(\theta)$ is the derivative of $\gamma$ with respect to $\theta$, is left invariant. For a vector field $Z(\gamma)$, the parallel transport to the left is given by $Z(\gamma)(\theta) = \gamma_1(\theta)Z(\gamma)(\theta)$. Let $L_{\gamma}(\gamma_1) = \gamma_1 \circ \gamma$ be the left translation. We define $\text{Ad}(\gamma) : V \to V$ by $\text{Ad}(\gamma)(u)(\theta) = \gamma'(\theta)u(\gamma(\theta))\gamma^{-1}(\theta))$. Then $\text{Ad}(\gamma^{-1})(u)(\theta) = \frac{u(\gamma(\theta))}{\gamma'(\theta)}$. For a rotation of angle $\phi$, $\theta + \phi = r_{\phi}(\theta)$. $\text{Ad}(r_{\phi})(u) = u(\theta + \phi)$. Let $u, v \in V$ and consider the vector fields $X_u(\gamma) = u \circ \gamma$ and $X_u(\gamma) = w \circ \gamma$; the bracket is given by $[X_u(\gamma), X_v(\gamma)] = [u, v] \circ \gamma$. For $Y_u(\gamma) = \gamma'(\theta)u(\theta)$, then $[Y_u(\gamma), Y_v(\gamma)] = \gamma'(\theta)[u, v]$.

3. The Hilbert transform

Let $J \cos k \theta = \sin k \theta$ and $J \sin k \theta = -\cos k \theta$, $k \geq 1$. The Hilbert transform $J$ allows to sort out the $\cos k \theta$, $\sin p \theta$, ... according to whether $k > p$ or $k < p$. With (2), we obtain

\[ J \cos k \theta, J \cos p \theta \] - $\cos k \theta, \cos p \theta = (p - k)\sin (p + k) \theta,
\]
\[ J \sin k \theta, J \cos p \theta \] - $\sin k \theta, \cos p \theta = (k - p)\cos (p + k) \theta
\]
and for $X = \sin k \theta$, or $X = \cos k \theta$ and $Y = \sin p \theta$, or $X = \cos p \theta$, with $p, k \geq 2$,

\[ [JX, JY] - [X, Y] = J([X, JY] + [JX, Y]). \tag{7} \]

Since $J^2 = -\text{Id}$, then (7) can be written $[X, JY] + [JX, Y] = J[X, Y] - J[JX, JY]$. Define

\[ A(X, Y) = J[X, Y] - [JX, JY] \quad \text{and} \quad B(X, Y) = J[X, Y] - [X, JY], \]

\[ \{X, Y\} = \frac{1}{2}([X, JY] + [JX, Y]) = \frac{1}{2}J([X, Y] - [JX, JY]). \tag{8} \]

When $X \neq Y$, we have $B(X, Y) = JB(X, JY)$ as well as $A(X, Y) = JA(X, JY)$ and $A(X, JY) + A(JX, Y) = 0$. Eq. (6) is equivalent to any of the two following $A(JX, JY) = A(X, Y)$ or $B(JX, JY) = B(X, Y)$. For $[,]$, we have the Jacobi identity

\[ \{X, [Y, Z]\} + \{Y, [Z, X]\} + \{Z, [X, Y]\} = 0 \tag{9} \]

and $\{JX, JY\} = J\{X, Y\}$. With (7), we deduce

\[ J([JX, JY] + [X, Y]) - ([JX, Y] - [X, JY]) = 2A(X, Y), \]
\[ J([JX, JY] + [X, Y]) + ([JX, Y] - [X, JY]) = 2B(X, Y). \tag{10} \]

4. The scalar product on $\text{diff}(S^1)/\text{su}(1, 1)$

With the scalar product (1), it was remarked in [3] that $\alpha(k) = \text{constant} \times (k^3 - k)$ when $\text{Ad}(h)$ is a unitary operator for any homographic transformation $h$. In fact, let $u_{2k}(\theta) = \cos(k \theta)$, $u_{2k+1}(\theta) = \sin(k \theta)$, the condition $\{u_p, u_q\} = 0 \forall p, q \geq 2$ and $u_1 = 1$, $\cos(\theta)$, $\sin(\theta)$ is equivalent to $(1 - k)\alpha(1 + k) + (2 + k)\alpha(k) = 0$ if $k \neq 2$ and $-\alpha(3) + 4\alpha(2) = 0$ which determines completely $\alpha(k) = \text{constant} \times (k^3 - k)$. In that case, for $m, n, p, j \in Z, (m - n)\alpha(p) + (n - p)\alpha(m) + (p - m)\alpha(n) = 0$ if $m + n + p = 0$ and
(j - k)(m + p)α(j + k) - (p - k)(m + j)α(p + k) = (j - p)[(k + j + 2)pα(k + j) + (p - k)α(j + p)] if \( m = k + j + p \).

**Lemma 4.1.** If \( \alpha(k) = \text{constant} \times (k^3 - k) \), for \( u, v, w \) in \( \text{diff}(S^1) \), we have

\[
\begin{align*}
\{ \sin k\theta & | \{ \cos j\theta, \cos p\theta \} \} + \{ \sin j\theta | \{ \cos p\theta, \cos k\theta \} \} + \{ \sin p\theta | \{ \cos k\theta, \cos j\theta \} \} = 0, \\
\{ \cos k\theta & | \{ \sin j\theta, \cos p\theta \} \} + \{ \cos j\theta | \{ \cos p\theta, \sin k\theta \} \} + \{ \sin p\theta | \{ \sin j\theta, \sin k\theta \} \} = 0.
\end{align*}
\]  

We verify (11) as follows. Let \( \delta^p \) be the Kronecker symbol, we add

\[
\begin{align*}
2\{ \sin k\theta & | \{ \cos j\theta, \cos p\theta \} \} = (j + p)\alpha(k)\delta^p_{j+p} + (j - p)\alpha(k)\delta^p_{j-p}, \\
2\{ \sin j\theta & | \{ \cos p\theta, \cos k\theta \} \} = (p - k)\alpha(j)\delta^p_{j+p} - (k + p)\alpha(j)\delta^p_{j-k}, \\
2\{ \sin p\theta & | \{ \cos k\theta, \cos j\theta \} \} = -\{ k + j \alpha(p) \delta^p_{j+p} + (k - j)\alpha(p)\delta^p_{j-k}, \\
2\{ \cos k\theta & | \{ \sin j\theta, \cos p\theta \} \} = (j + p)\alpha(k)\delta^p_{j+p} - (j - p)\alpha(k)\delta^p_{j-p}, \\
2\{ \cos j\theta & | \{ \cos p\theta, \sin k\theta \} \} = (k + j)\alpha(p)\delta^p_{j+p} + (j - k)\alpha(p)\delta^p_{j-k}, \\
2\{ \sin p\theta & | \{ \sin j\theta, \sin k\theta \} \} = (k + j)\alpha(p)\delta^p_{j+p} - (k - j)\alpha(p)\delta^p_{j-k}.
\end{align*}
\]

From (11), we deduce that \( \{ u | \{ \Gamma(v), \Gamma(w) \} \} + \{ v | \{ \Gamma(w), \Gamma(u) \} \} + \{ w | \{ \Gamma(u), \Gamma(v) \} \} = 0 \) or equivalently (3) is true for \( u, v, w \) of the form \( \cos k\theta \) or \( \sin j\theta \). We use the linearity in each variable to prove (3) in its generality.

From (3), we obtain more identities, let \( \{ u | v, u \} = \frac{1}{2}(\{ u | v, u \} + \{ v | w, u \} + \{ w | u, v \} \), then

\[
\begin{align*}
\{ u | v, w \} + \{ v | w, u \} + \{ w | u, v \} &= \frac{1}{2}(\{ u | v, w \} + \{ v | w, u \} + \{ w | u, v \}), \\
\{ u | v, w \} = \frac{1}{2}(\{ u | v, w \} + \{ v | w, u \} + \{ w | u, v \}).
\end{align*}
\]  

5. The Levi-Civita’s transfer field and related fields

We follow [12, vol. 1, p. 160 and vol. 2, p. 201], [7] and [2, p. 452]. For \( u, v, w \in \text{diff}(S^1) \), we define \( \Gamma_{\text{LC}}(u, v, w) \) by (4). We denote \( \Gamma_{\text{LC}}(u, v, w) = (\Gamma_{\text{LC}}(u | v, w) = (\Gamma_{\text{LC}}^w_{uv}) \). Since \( \Gamma_{\text{LC}}(u, v, w) \) is antisymmetric in \( u, v, w \), the adjoint \( \Gamma_{\text{LC}}^* \) satisfies \( \Gamma_{\text{LC}}^* = -\Gamma_{\text{LC}}^* \). With (3), we deduce that the connection \( \Gamma_{\text{LC}} \) preserves the complex structure, \( \Gamma_{\text{LC}}(u, v, w) = \Gamma_{\text{LC}}(Ju, Jv, Jw) \). The transfer field \( \Gamma_{\text{LC}} \) is the half sum of two antisymmetric transfer fields

\[
\Gamma_{\text{LC}} = \frac{1}{2}(\Gamma_3 + \Gamma_3), \quad \text{with } \Gamma_3(u, v, w) = \{ [u, w] | \pi v \}, \quad \Gamma_3(u, v, w) = \{ [w, v] | \pi u \} - \{ [u, v] | \pi w \}.
\]

For \( \Gamma = \Gamma_3, \Gamma_3, \Gamma_3(v)J \neq J\Gamma(v) \), we obtain \( \Gamma_{\text{LC}}^w = \Gamma(u, v, w) \) and define

\[
\begin{align*}
\Gamma_1(u, v, w) &= \{ [u, v] | \pi w \} = (\Gamma_1^w)_{uv}, \\
\Gamma_2(u, v, w) &= \{ [v, w] | \pi u \} = (\Gamma_2^w)_{uv}, \\
\Gamma_4(u, v, w) &= \{ [w, u] | \pi v \} + \{ [w, v] | \pi u \} = (\Gamma_4^w)_{uv}.
\end{align*}
\]

Then \( \Gamma_{\text{LC}} = \frac{1}{2}(\Gamma_4 - \Gamma_1) \) and \( \Gamma_3 = -(\Gamma_1 + \Gamma_3) \). Another connection also preserves the complex structure, see [6],

\[
\Gamma_{\text{LC}}(u, v, w) = \Gamma_{\text{LC}}(Ju, Jv, Jw) = (\Gamma_{\text{LC}}(u | v, w) = (\Gamma_{\text{LC}}^w)_{uv}.
\]  

\[
\Gamma_{\text{LC}}(u, v, w) = \Gamma_{\text{LC}}(Ju, Jv, Jw) = (\Gamma_{\text{LC}}(u | v, w) = (\Gamma_{\text{LC}}^w)_{uv}.
\]
Let \( A(v, w) \) and \( B(v, w) \) as in (8). Then
\[
(B(v, w) | u) - (B(v, u) | w) = (J v | [w, u]) + ([J w, v] | u) - ([J u, v] | w).
\]
\[
(A(v, w) | u) - (A(v, u) | w) = (\Gamma_A(J v) u | w),
\]
\[
J \Gamma_A(J v) u + \Gamma_A(v) u = 2 B(u, v).
\]

To prove the last identity, we remark with (3) that
\[
(J \Gamma_A(J v) u | w) - (J \Gamma_A(v) u | w) = 2 B(u, v) | w).
\]
With (3), we obtain that
\[
(\Gamma_A(J v) u | w) - (\Gamma_A(v) u | w) = 2 (J B(u, v) | w).
\]

6. Composition of transfer fields: expression with symbols

The following operators \( \Gamma_{e_j} \) are diagonal in the basis \( |c_m, s_n\rangle \) and the series \( \sum_{j \geq 2} E_j \) converge,
\[
(\Gamma^2_{e_j} + \Gamma_{e_j}) c_m = -\frac{1}{2} A_{mj}^2 + A^2_{j m} c_m,
\]
\[
(\Gamma^2_{e_j} + \Gamma_{e_j}) c_m = -\frac{1}{2} A_{mj}^2 + A^2_{j m} c_m,
\]

For \( j \geq 2 \), let \( \epsilon_j = c_j(\theta) = \frac{\cos j \theta}{\sqrt{2}} \) and \( \epsilon_{j+1} = s_j(\theta) = \frac{\sin j \theta}{\sqrt{2}} \). If \( \Gamma = \Gamma_{e_j} \), \( \Gamma_{e_k} \) or any \( \Gamma \) as above, we denote \( I_{e_j e_k} = (\Gamma_{e_j}, e_k, e_j) = (\Gamma_{e_j})_{e_k} e_j \). Let \( \delta^k_\ell \) be the Kronecker symbol. The symbol
\[
A_{j k p} = \delta^{k+j} p (j + k) \sqrt{\frac{\alpha(p)}{\alpha(k)}} \quad \text{for} \quad j, k, p \geq 2,
\]
is convenient to calculate in a systematic way the composition of \( \Gamma \)’s,
\[
\frac{1}{2} [A_{j k p} + A_{p k j}] = \frac{1}{2} \left( [c_p, s_j] c_k \right) + \left( [c_p, s_k] s_j \right) - \left( [s_j, c_k] c_p \right) = (\Gamma_{e_j}^p c_k)_{e_k} = -(\Gamma_{e_k}^p c_j)_{e_j},
\]
\[
\frac{1}{2} [A_{j k p} - A_{p k j}] = \frac{1}{2} \left( [c_p, s_j] s_k \right) - \left( [c_p, s_k] c_j \right) + \left( [s_j, c_k] c_p \right) = -(\Gamma_{e_j}^p s_k)_{e_k},
\]

The following operators \( E_j \) are diagonal in the basis \( |c_m, s_n\rangle \) and the series \( \sum_{j \geq 2} E_j \) converge,
\[
\begin{align*}
\Gamma_{4}(c_j)\Gamma_{1}(c_j) + \Gamma_{3}(s_j)\Gamma_{1}(s_j) = & \left[\frac{1}{2}(A_{jm}A_{jr} - A_{jm}A_{jr}) + \frac{1}{2}(A_{mjr} + A_{mrj})\right]c_m, \\
\Gamma_{4}(c_j)\Gamma_{2}(c_j) + \Gamma_{3}(s_j)\Gamma_{2}(s_j) = & \left[\frac{1}{2}(A_{jr}A_{jm} - A_{jr}A_{jm}) + \frac{1}{2}(A_{mjr} - A_{mrj})\right]c_m, \\
\Gamma_{1}(c_j)\Gamma_{2}(c_j) + \Gamma_{1}(s_j)\Gamma_{2}(s_j) = & \left[-\frac{1}{2}(A_{jr}A_{jm} + A_{jm}A_{jr}) + \frac{1}{2}(A_{mrj} - A_{mjr})\right]c_m, \\
\Gamma_{LC}(c_j)\Gamma_{2}(c_j) + \Gamma_{LC}(s_j)\Gamma_{2}(s_j) = & \left[\frac{1}{2}A_{jr}A_{jm} - A_{rjm}^2\right]c_m.
\end{align*}
\]

The operator \( \Lambda \) defined by (5) is a bounded operator with the metric \( \alpha(k) = k^3 - k \),

\[
2\Lambda c_m = -\sum_{j \geq 2} A_{mjr}A_{mrj}c_m = -\sum_{j \geq 2} \frac{(2m - j)(m + j)}{\alpha(m)}1_{m \geq j + 2}c_m = - \frac{13}{6} + \text{a finite number of terms in } \frac{1}{m}, \tag{21}
\]

\( \Lambda \) is related to the tangent processes introduced in the works \([7,9]\). Among all the Driver’s connections, see \([8]\), \( \Gamma_{LC} \) and \( \Gamma_{A} \) are the only ones for which \( \sum_{j \geq 2} \Gamma^2(c_j) + \Gamma^2(s_j) \) converge. The series \( \sum_{j \geq 2} \Gamma^2(c_j) + \Gamma^2(s_j) \) converge for \( \Gamma_{LC} \) and \( \Gamma_{A} \), it diverge for \( \Gamma = \Gamma_{3}, \Gamma_{5} \) and \( \Gamma_{1}, \Gamma_{2} \).

References