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Partial Differential Equations

Optimal distributed-control of vortices in Navier–Stokes flows ☆

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Abstract

Optimal control of time-dependent fluid-flow governed by the incompressible Navier–Stokes model is considered. A cost functional based on a local dynamical systems characterization of vortices is investigated. The resulting functional is a non-convex function of the velocity gradient tensor. The optimality systems based on a Lagrangian formulation and adjoint equations describing first order necessary optimality conditions is provided. The first and the second derivative of the cost functional with respect to the control are derived. ***To cite this article: S. Chaabane et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Contrôle optimal des vorticités dans des écoulement de Navier–Stokes. On considère un problème de contrôle optimal d'un écoulement gouverné par les équations de Navier–Stokes dynamiques. On considère une fonctionnelle coût basée sur les systèmes dynamiques locaux caractérisant les tourbillions. La fonctionnelle traitée est non-convexe par rapport au tenseur du gradient des vitesses. Les systèmes d'optimalité basés sur la formulation du Lagrangien et le problème adjoint décrivant les conditions d'optimalité nécessaires de premier ordre sont fournis. Le gradient ainsi que la seconde dérivée de la fonctionnelle coût par rapport au contrôle sont établis. ***Pour citer cet article : S. Chaabane et al., C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

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On s'interesse à un problème de contrôle optimal des fluides gouvernés par le modèle de Navier–Stokes incompressible formulé dans un cylindre temps-espace $Q = (0, T) \times \Omega$, $T > 0$ où Ω est un domaine bidimensionnel d'un seul côté de sa frontière Γ de classe C^2. Le problème est donné par les Éqs. (4), Le but principal consiste à établir un modèle mathématique lié à la nouvelle formulation de réduction des tourbillions dans un fluide incompressible, (voir [3]). La fonctionnelle coût à minimiser dans l'ensemble U des contrôles admissibles u est la suivante $J(y, u) = J_1(y) + \beta J_2(u)$.

Dans plusieurs cas, un choix propre de J_2 est donné par l'Éq. (1), où T et $\Omega_c \subset \Omega$ sont respectivement l'horizon et le support du contrôle.

La fonctionnelle J_1 est déduite d'une analyse locale basée sur les systèmes dynamiques caractérisés par l'écoulement $y(x)$. Différentes études suggèrent que les tourbillions sont souvent localisés dans les régions où le $\det \nabla(y) > 0$; ce qui nous amène à faire le choix donné par l'Éq. (2), où Ω_0 est la région où l'on désire de des tourbillions et h est une fonction positive adéquate.

Notons que J_1 est une fonction non convexe par rapport au tenseur des vitesses y du fluide.

Le problème de contrôle qu'on considère consiste à optimiser la fonctionnelle coût J sous la contrainte des équations de Navier–Stokes dont le contrôle est le terme source (Éq. (5)). Dans ce travail, on montre les résultats suivants :

Théorème 0.1. *Le problème de contrôle optimal* (5) *admet une solution* (y, p, u).

On établit la dérivabilité de l'état de l'écoulement par rapport au contrôle en se basant sur le théorème des fonctions implicites au sens faible, [1].

Théorème 0.2. *La solution y du problème de Navier–Stokes est différentiable par rapport au contrôle u dans U. La dérivée de l'état en un point u dans une direction δu satisfait les Éqs.* (6).

On donne le système d'optimalité de premier ordre associé aux solutions du problème de contrôle optimal à travers le Lagrangien associé. On fournit le gradient de la fonctionnelle par rapport au contrôle.

Théorème 0.3. *Le gradient de la fonctionnelle coût en u dans une direction δu dans U est donné par l'Éq.* (9)*, où $\hat{J}$ est la réduite de J donnée par l'expression* (8) *et ξ est l'état adjoint calculé à partir du système d'optimalité.*

À la fin de cette Note, on donne la dérivée seconde de la fonctionnelle par rapport au contrôle.

Proposition 0.4. *La dérivée seconde de la fonctionnelle réduite en u est donnée par l'Éq.* (10).

1. Introduction

This work deals with optimal control related to fluid flow. The motion of the fluid is governed by the incompressible Navier–Stokes model with non-homogeneous Dirichlet conditions. The aim of this work is to provide a mathematical study linked to the novel optimal control formulation for the reduction and possibly extinction of vorticities in an incompressible fluid introduced in [3]. Optimal control is based on minimization of a cost functional $J(y, u) = J_1(y) + \beta J_2(u)$ over a set U of admissible controls u, where $y = y(t, x)$ denotes the velocity vector of the fluid at time $t > 0$ and location x in the spatial domain Ω and $\beta > 0$ stands for the control costs. The controls u can be body forces or an action like blowing or sucking along the boundary of Ω. Alternatively, the control action on the fluid can be enforced in a more indirect manner like heating or cooling. In many cases a proper choice for J_2 is given by

$$J_2(u) = \frac{1}{2}\int_0^T \int_{\Omega_c} \left|u(t,x)\right|^2 \mathrm{d}x\,\mathrm{d}t \tag{1}$$

where T is the control horizon and $\Omega_c \subset \overline{\Omega}$ the support of the controller.

In [3] a definition of J_1 is suggested, serving the purpose of penalizing vorticity, by connecting it to vortex via a local analysis based on dynamical systems characterized by the flow $y(x)$. Several propositions which are different in dimension 3 coincide in dimension 2 and suggest to link vorticity to region where $\det(\nabla y) > 0$. This leads to the choice

$$J_1(y) = \alpha \int_0^T \int_{\Omega_0} h\big(\det\big(\nabla y(t,x)\big)\big)\,\mathrm{d}t\,\mathrm{d}x \tag{2}$$

where Ω_0 is the region in which it is desired to suppress vortices, and h is a positive function whenever its argument is positive. The choice of the function h is motivated by the fact that the period lengths of the trajectories generated by the velocity $y(x)$, are closed curves, decrease as $\det(\nabla y)$ increases. This allows us to choose h as a monotonically increasing function of time. To allow differentiability of the cost functional J, h is chosen as C^1-function. This results in a possible choice for h given by

$$h(s) = \begin{cases} 0 & \text{if } s < -\delta, \\ \dfrac{s^2}{2\delta} + s + \dfrac{\delta}{2} & \text{if } -\delta \leqslant s \leqslant 0, \\ s + \dfrac{\delta}{2} & \text{if } 0 \leqslant s \end{cases} \tag{3}$$

for fixed and small $\delta > 0$.

2. Navier–Stokes equations

Let Ω be a bounded spatial two-dimensional domain locally on one side of its C^2-boundary Γ. By (y_1, y_2) we denote the velocity of the fluid at $x = (x_1, x_2)$ and by p its pressure. The controlled time-dependent Navier–Stokes equations on the time-space cylinder $Q = (0, T) \times \Omega$, $T > 0$ are given by

$$\begin{cases} y_t - \dfrac{1}{Re}\Delta y + (y\cdot\nabla)y + \nabla p = \mathcal{B}u & \text{in } Q, \\ \operatorname{div}(y) = 0 & \text{in } Q, \\ y = g & \text{on } \Sigma = (0,T)\times\Gamma, \\ y(0,x) = y_0(x) & \text{in } \Omega \end{cases} \tag{4}$$

where $u \in U$ is the control variable, U is the space of controls. $\mathcal{B} \in \mathcal{L}(U, L^2(Q))$ and $y_0 \in H$. Further Re is the Reynolds number. The function $\mathcal{B}u \in L^2(Q)$ represents a volume force. If the control acts on a subset Ω_c of Ω, then $\mathcal{B}$ is the extension by zero operator: $\mathcal{B}u = u$ in Ω_c and $\mathcal{B}u = 0$ in $\Omega\backslash\Omega_c$. The function g satisfying $\int_\Sigma g\cdot n = 0$ is fixed in W_Σ and n is the outward normal on Γ.

2.1. Functional setting

Let us introduce the functional setting arising from the previous equations,

$$H = L^2(\Omega)^2, \quad V = H^1(\Omega)^2, \quad L_0^2(\Omega) = L^2(\Omega)/\mathbb{R},$$
$$\mathcal{W} = \{v \in L^2(V):\ v_t \in L^2(V^*)\}, \quad Z := L^2(V) \times H, \quad X = \mathcal{W} \times U,$$

$\mathcal{W}$ is endowed with the norm $\|v\|_{\mathcal{W}} = (|v|^2_{L^2(V)} + |v_t|^2_{L^2(V^*)})^{1/2}$,

$$H^{2,1}(Q) = \{v \in L^2(V \cap H^2(\Omega)^2);\ v_t \in L^2(H)\}.$$

We set $\langle\cdot,\cdot\rangle := \langle\cdot,\cdot\rangle_{L^2(V^*),L^2(V)}$ with V^* denoting the topological dual space of V. Here $L^2(V)$ is an abbreviation for $L^2(0,T;V)$ and similarly $L^2(V^*) = L^2(0,T;V^*)$. U denotes the Hilbert space of controls which is identified with its dual. Let $W_\Sigma = \{g = \tau\hat{g}:$ with $\hat{g} \in \mathcal{W}\}$ the set of the admissible inhomogeneities for Dirichlet boundary conditions (see [4]).

Definition 2.1. Set $\hat{e}(y,u) = (y_t - \frac{1}{Re}\Delta y + (y\cdot\nabla)y - \mathcal{B}u,\ y(0) - y_0)$ where $\hat{e}$ is defined from $\mathcal{W} \times U$ to $L^2(V^*) \times H$.

Theorem 2.2. *For $\mathcal{B}u$ any given data in $L^2(Q)$ and g in W_Σ. Then the associated Navier–Stokes problem has a unique solution (y,p) in the space $\mathcal{W} \times L^2(0,T;L_0^2(\Omega))$ (see* [2,4]).

2.2. Cost functional

For positive scalars α and β we define

$$J(y,u) = \alpha \int_0^T \int_{\Omega_0} h(\det\nabla y)\,\mathrm{d}x\,\mathrm{d}t + \frac{\beta}{2}\|u\|_U^2$$

where $\Omega_0 \subseteq \Omega$ with Γ_0 its boundary and set $\overline{\Gamma} = \Gamma_0 \backslash \Gamma$. Clearly, J is bounded from below by zero, and $J(y,u) \to \infty$ as $\|u\|_U \to \infty$ for every y in $\mathcal{W}$.

3. The optimal control problem

The optimal control problem that we consider has the form

$$\begin{cases} \min J(y,u) \text{ over } (y,u) \in \mathcal{W} \times U \\ \text{subject to (4)} \end{cases} \tag{5}$$

Theorem 3.1. *The optimal control problem* (5) *has a solution (y,p,u) in $\mathcal{W} \times L^2(L_0^2(\Omega)) \times U$.*

Proposition 3.2. *The cost functional J has at least one admissible minimum (y,u) in $\mathcal{W} \times U$.*

Proposition 3.3. *Let S_p be the set of all minima of the cost functional J subject to* (4)*. Then, S is weakly compact with respect to the topology induced by the space $\mathcal{W} \times U$.*

4. First order-optimality system

4.1. Differentiability of the state with respect to the control

In order to prove the differentiability of y with respect to the control u, we cannot use the classical implicit function theorem, since it requires strong differentiability results in H^{-1} for our applications. We therefore use the weak implicit function theorem (see [1]).

Theorem 4.1. *The solution y to the Navier–Stokes problem is differentiable with respect to the control u in U. Its derivative at a point u in a direction δu satisfies*

$$\hat{e}_y(x)y^1(u)\cdot\delta u = -\hat{e}_u(x)\cdot\delta u$$

and

$$\begin{cases} y^1_t - \dfrac{1}{Re}\Delta y^1 + (y^1\cdot\nabla)y + (y\cdot\nabla)y^1 = \mathcal{B}\delta u & \text{in } Q,\\ \operatorname{div}(y^1) = 0 & \text{in } Q,\\ y^1 = 0 & \text{on } \Sigma,\\ y^1(0,x) = 0 & \text{in } \Omega. \end{cases} \tag{6}$$

4.2. Lagrangian

Let us introduce the Lagrangian associated to the cost functional J subject to the weak formulation of the Navier–Stokes problem as a constraint: $\mathcal{L}: \mathcal{W}\times L^2(L^2_0(\Omega))\times U\times\mathcal{W}\times L^2(L^2_0(\Omega))\to\mathbb{R}$ defined by

$$\mathcal{L}(z,q,v,\varphi,\omega) = J(z,v) + \int_Q \left(z_t - \frac{1}{Re}\Delta z + Dz\cdot z + \nabla q - \mathcal{B}v\right)\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_Q \omega\,\operatorname{div}(z)\,\mathrm{d}x\,\mathrm{d}t.$$

4.3. First order-optimality conditions

Lemma 4.2. *The mapping $y\to J_1(y)$ is differentiable from $\mathcal{W}$ to $\mathbb{R}$.*

Accordingly, we establish the optimality system associated to problem (5).

Theorem 4.3. *Let (y^*, p^*, u^*) be a solution to the problem* (5) *and (ξ^*, π^*) be the adjoint variables. Then the optimal system satisfied by $(y^*, p^*, u^*, \xi^*, \pi^*)$ is given by*

$$\begin{cases} y^*_t - \dfrac{1}{Re}\Delta y^* + (y^*\cdot\nabla)y^* + \nabla p^* = \mathcal{B}u^* & \text{in } Q,\\ \operatorname{div}(y^*) = 0 & \text{in } Q,\\ y^* = g & \text{on } \Sigma,\\ y^*(0,x) = y_0(x) & \text{in } \Omega,\\ -\xi^*_t - \dfrac{1}{Re}\Delta\xi^* - (\xi^*\cdot\nabla)y^* + (\nabla y^*)^T\xi^* + \nabla\pi^* = R(y^*) & \text{in } Q,\\ \operatorname{div}(\xi^*) = 0 & \text{in } Q,\\ \xi^* = 0 & \text{on } \Sigma,\\ \xi^*(T,x) = 0 & \text{in } \Omega,\\ \beta u^* - \mathcal{B}^*\xi^* = 0 & \text{in } U. \end{cases} \tag{7}$$

Here, $\mathcal{B}^*: L^2(Q) \to U$ denotes the adjoint of the operator $\mathcal{B}$, $((\nabla y)^T \xi)_i = \Sigma_j (\frac{\partial y_i}{\partial xi})\xi_i$ and the right-hand side $R(y)$ is given by

$$R(y) = -\alpha \begin{pmatrix} -\operatorname{curl}(h(\det \nabla y)\nabla y_2) + \chi_{[0,T]\times\overline{\Gamma}_0} h'(\det \nabla y)(\partial_2 y_2 - \partial_1 y_2) \\ \operatorname{curl}(h(\det \nabla y)\nabla y_1) - \chi_{[0,T]\times\overline{\Gamma}_0} h'(\det \nabla y)(\partial_2 y_2 - \partial_1 y_2) \end{pmatrix}$$

where $\chi_{[0,T]\times\overline{\Gamma}_0}$ denotes the characteristic function of the set $[0,T] \times \overline{\Gamma}_0$. We notice that $R(y)$ lies in $\mathcal{W}^*$.

4.4. Derivatives of the cost functional with respect to the control

For second order methods applied to optimal control problems (5) can be expressed as the reduced problem

$$\min \hat{J}(u) = J\big(y(u), u\big) \quad \text{over } u \in U. \tag{8}$$

Theorem 4.4. *The gradient of the reduced functional at u in direction δu in U is given by*

$$\big\langle \hat{J}'(u), \delta u \big\rangle_U = \langle \beta u, \delta u \rangle_U - \int_Q (\xi, \mathcal{B}\delta u), \tag{9}$$

where ξ is computed from the optimality system.

Let $x = (y, u)$. We introduce the Lagrangian $L: X \times Z \to \mathbb{R}$ with $L(y, u, \lambda) = J(y, u) + \langle \hat{e}(y, u), \lambda \rangle_{Z^*, Z}$ and let the mapping $T: X \to \mathcal{W} \times U$ with $T^*(x) = (-\hat{e}_y^{-1}(x)\hat{e}_u(x), \mathrm{Id}_U)$ and range $T(x) = \ker(\hat{e}_x(x))$.

Proposition 4.5. *The second derivative of the reduced cost functional at u is given by*

$$\hat{J}''(u) = T^*(x) L_{xx}(x, \lambda) T(x) \tag{10}$$

where

$$L_{xx} = \begin{pmatrix} J_{yy} + \langle \hat{e}_{yy}, \lambda \rangle & 0 \\ 0 & J_{uu} \end{pmatrix}.$$

5. Conclusion

We have investigated a cost functional involving the velocity gradient tensor of the fluid for the reduction of vortices in optimal control based formulations of vortex reduction strategies. The optimality system, the first and the second derivative of the cost functional with respect to the control are characterized. The results encourage further analysis of the proposed techniques including boundary control, three-dimensional problems and systems where the fluid interacts with its environment.

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