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C. R. Acad. Sci. Paris, Ser. I 341 (2005) 79–84
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Group Theory

Normal, unipotent subgroup schemes of reductive groups

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Received and accepted 19 May 2005

Available online 11 July 2005

Presented by Jean-Pierre Serre

Abstract

Let G be a reductive group over a field k of characteristic p. Let k^{sep} be a separable closure of k. If $p \neq 2$, there exists a linear representation of G that is faithful and semisimple; moreover, any unipotent, normal subgroup scheme of G is trivial. For $p = 2$, these two properties hold if and only if $G_{k^{\mathrm{sep}}}$ has no direct factor that is isomorphic to $\mathbf{SO}_{2n+1}$ for some $n \geqslant 1$. ***To cite this article: A. Vasiu, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Résumé

Sous-schémas en groupes unipotents normaux des groupes réductifs. Soit G un groupe réductif sur un corps k de caractéristique p. Soit k^{sep} une clôture séparable de k. Si $p \neq 2$, il existe une représentation linéaire de G qui est à la fois fidèle et semi-simple ; de plus, tout sous-schéma en groupes unipotent normal de G est trivial. Si $p = 2$, ces deux propriétés ne sont vraies que si $G_{k^{\mathrm{sep}}}$ n'a aucun facteur direct isomorphe à un groupe $\mathbf{SO}_{2n+1}$, $n \geqslant 1$; en particulier, elles sont fausses pour le groupe $\mathbf{SO}_3 = \mathbf{PGL}_2$. ***Pour citer cet article : A. Vasiu, C. R. Acad. Sci. Paris, Ser. I 341 (2005).***

Version française abrégée

Soit G un groupe algébrique réductif (donc connexe et lisse) sur un corps k de caractéristique $p > 0$. Nous nous intéresserons aux deux propriétés suivantes (le texte anglais en contient deux autres) :

S_1. *Tout sous-schéma en groupes unipotent de G qui est normal dans G est trivial.*

(Pour la définition de « unipotent » dans le contexte des schémas en groupes, voir [2, Vol. II, Exp. XVII, 3.5].)

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doi:10.1016/j.crma.2005.05.014

S_2. *Il existe une représentation linéaire* $\rho : G \to \mathbf{GL}_V$ *de* G *qui est à la fois fidèle et semi-simple.*

(L'adjectif « fidèle » est pris au sens de la théorie des schémas ; il signifie que le noyau de ρ est trivial, ou encore que ρ définit un isomorphisme de G sur un sous-schéma en groupes fermé de $\mathbf{GL}_V$, cf. [2, Vol. I, Exp. VI_A, 5.4].)

Il n'est pas difficile de voir que $S_1 \Leftrightarrow S_2$. De plus :

Théorème 0.1. *Si* $p \neq 2$, *les propriétés* S_1 *et* S_2 *sont vraies quel que soit* G.

Théorème 0.2. *Supposons* $p = 2$.

(i) *Si* $G = \mathbf{SO}_{2n+1}$, $n \geqslant 1$, *les propriétés* S_1 *et* S_2 *ne sont pas vraies.*
(ii) *Inversement, si* S_1 *et* S_2 *ne sont pas vraies pour* G, *alors, après extension des scalaires,* G *se décompose en un produit direct dont l'un des facteurs est un groupe* $\mathbf{SO}_{2n+1}$ *avec* $n \geqslant 1$.

Corollaire 0.3. *Si* $p = 2$, *le groupe* $\mathbf{PGL}_2 = \mathbf{SO}_3$ *n'a pas de représentation linéaire fidèle semi-simple.*

Théorème 0.4. *Supposons* $p = 2$. *Le groupe* G *possède un plus grand sous-schéma en groupes* U *unipotent et normal. Le groupe* U *est infinitésimal de hauteur* 1, *et isomorphe à un produit de groupes de type* $\boldsymbol{\alpha}_2$. *Le quotient* G/U *jouit des propriétés* S_1 *et* S_2.

(Le groupe U joue donc le rôle d'un « radical unipotent schématique ».)

Indications sur les démonstrations des théorèmes 0.1, 0.2 , 0.4

Le point essentiel consiste à déterminer les sous-groupes unipotents normaux de G qui sont infinitésimaux de hauteur 1. Un tel sous-groupe U correspond à une p-sous-algèbre de Lie $\mathfrak{u}$ de $\mathrm{Lie}(G)$ qui est p-nilpotente et stable par l'action de G sur $\mathrm{Lie}(G)$. Si l'on étend les scalaires pour déployer G, on peut écrire $\mathrm{Lie}(G)$ à la façon habituelle

$$\mathrm{Lie}(G) = \mathrm{Lie}(T) \bigoplus_{\alpha \in R} k X_\alpha$$

où T est un tore maximal et les α parcourent le système de racines R de (G, T). Comme $\mathfrak{u}$ est stable par l'action de T, et que $\mathfrak{u} \cap \mathrm{Lie}(T) = 0$, on voit que $\mathfrak{u}$ est somme directe des kX_α, pour α parcourant une partie $R_\mathfrak{u}$ de R. Comme $\mathfrak{u}$ est stable par le normalisateur de T dans G, $R_\mathfrak{u}$ est stable par le groupe de Weyl de R. Si $\alpha \in R_\mathfrak{u}$, on a $-\alpha \in R_\mathfrak{u}$, d'où $[X_\alpha, X_{-\alpha}] = 0$ puisque $\mathfrak{u} \cap \mathrm{Lie}(T) = 0$. On obtient là une condition très restrictive sur $R_\mathfrak{u}$. Il n'est pas difficile d'en déduire que, si G est simple, et $R_\mathfrak{u}$ non vide, alors $p = 2$, G est de type B_n adjoint, et $R_\mathfrak{u}$ est formé des racines courtes, auquel cas U est abélien et G/U est de type C_n simplement connexe. Les théorèmes 0.1, 0.2 et 0.4 s'en déduisent facilement.

1. Introduction

Let $p \in \mathbf{N}$ be a prime. Let k be a field of characteristic p. Let k^{sep} be a separable closure of k. Let V be a finite dimensional vector space over k. Let $\widetilde{G}$ be an affine group scheme of finite type over k. The Lie algebra $\mathrm{Lie}(\widetilde{G})$ of $\widetilde{G}$ is a left $\widetilde{G}$-module via the adjoint representation of $\widetilde{G}$. A linear representation $\tilde{\rho} : \widetilde{G} \to \mathbf{GL}_V$ is called *faithful* if it is a closed embedding (equivalently if the schematic kernel $\mathrm{Ker}(\tilde{\rho})$ as defined in [2, Vol. I, Exp. VI_A, 5.4], is trivial). An affine group scheme U of finite type over k is called *unipotent* if any one of the following three equivalent conditions holds (see [2, Vol. II, Exp. XVII, 3.5]):

(i) any nonzero linear representation of U has a nonzero fixed vector;
(ii) the group scheme U is isomorphic to a subgroup scheme of the unipotent radical of a Borel subgroup of the $\mathbf{GL}_n$ group over k (here $n \geqslant 1$);
(iii) the group scheme U has a composition series whose factors are $\mathbf{G}_a$, $\boldsymbol{\alpha}_p$, or a form of $(\mathbf{Z}/p\mathbf{Z})^s$ with $s \geqslant 1$.

From (iii) we get that the unipotent group schemes are stable under subgroups, quotient groups, extensions, and field extensions.

Let G be a reductive group over k. So G is smooth, connected, affine and has no nontrivial, normal, unipotent, smooth subgroup. In this Note we study normal, unipotent subgroup schemes of G in connection with the following four natural statements on G.

S_1. *Every normal, unipotent subgroup scheme of G is trivial.*
S_2. *There exists a faithful linear representation $\rho : G \hookrightarrow \mathbf{GL}_V$ that is semisimple.*
S_3. *Suppose $G = \widetilde{G}/U$, where $\widetilde{G}$ is assumed to be a smooth, connected group over k and U is a unipotent, normal subgroup scheme of $\widetilde{G}$. Let $\tilde{\rho} : \widetilde{G} \hookrightarrow \mathbf{GL}_V$ be a faithful linear representation. Let $0 = V_0 \subset V_1 \subset \cdots \subset V_m = V$ be a filtration of V stable under $\widetilde{G}$ and such that the quotients V_i/V_{i-1} are semisimple $\widetilde{G}$-modules, $i \in \{1, \ldots, m\}$. The group U acts trivially on $\mathrm{gr}(V) := \bigoplus_{i=1}^m V_i/V_{i-1}$ and thus we have a linear representation $\tilde{\rho}_{\mathrm{ss}} : G \to \mathbf{GL}_{\mathrm{gr}(V)}$. Then $\tilde{\rho}_{\mathrm{ss}}$ is faithful.*
S_4. *Let $f : G \to G'$ be a homomorphism, where G' is another reductive group over k. Let T and T' be maximal tori of G and G' (respectively) such that $f(T) \subset T'$. Let $f_T : T \to T'$ be the homomorphism defined by f. If f_T is an embedding, then f is also an embedding.*

(So if f is an isogeny such that f_T is an isomorphism, then f is an isomorphism.)

It is not difficult to check that statements S_1 to S_4 are equivalent (see Subsection 3.1).

Theorem 1.1. *If $p \neq 2$, then all four statements S_1 to S_4 are true.*

Theorem 1.2. *Let $p = 2$. Then statements S_1 to S_4 are false in general. More precisely, statement S_i (with $i \in \{1, 2, 3, 4\}$) is false if and only if $G_{k^{\mathrm{sep}}}$ has a direct factor that is isomorphic to $\mathbf{SO}_{2n+1}$ for some $n \geqslant 1$.*

So if $p = 2$, then the group $\mathbf{SO}_{2n+1}$ (with $n \geqslant 1$) has no faithful linear representation that is semisimple; in particular, this applies to $\mathbf{PGL}_2 = \mathbf{SO}_3$. Statements S_1 and S_4 are implicit in both [3] and [5]. In Section 2 we study pairs (G, U), where G is as above and U is a nontrivial, normal, unipotent subgroup scheme of G. In Subsection 2.2 we show that if such a pair (G, U) exists, then $p = 2$ and $U \xrightarrow{\sim} \boldsymbol{\alpha}_2^{2l}$, where $l \geqslant 1$. In Section 3 we prove Theorems 1.1 and 1.2.

2. Classification results

We first include a lemma on certain G-submodules of $\mathrm{Lie}(G)$. In Subsection 2.1 we take $p = 2$ and we recall the classical isogeny $\mathbf{SO}_{2n+1} \to \mathbf{Sp}_{2n}$ whose kernel is isomorphic to $\boldsymbol{\alpha}_2^{2n}$. In Subsection 2.2 we prove a result that is the very essence behind the four statements $S_1, \ldots, S_4$ and that classifies all normal, unipotent subgroup schemes of G.

Lemma 2.1. *Let T be a maximal torus of G. Let $\mathfrak{u}$ be a nonzero G-submodule of $\mathrm{Lie}(G)$ such that $\mathfrak{u} \cap \mathrm{Lie}(T) = 0$. Then $p = 2$ and $G_{k^{\mathrm{sep}}}$ has a direct factor G_1 which is isomorphic to $\mathbf{SO}_{2n+1}$ (for some $n \geqslant 1$) and for which we have $\mathrm{Lie}(G_1) \cap \mathfrak{u} \otimes_k k^{\mathrm{sep}} \neq 0$.*

Proof. We can assume $k = k^{\mathrm{sep}}$. So T is split, cf. [2, Vol. III, Exp. XXII, 2.4]. Let

$$\mathrm{Lie}(G) = \mathrm{Lie}(T) \bigoplus_{\alpha \in R} V_\alpha$$

be the root decomposition relative to T. As $\mathfrak{u} \cap \mathrm{Lie}(T) = 0$ and as T normalizes $\mathfrak{u}$, there exists a subset $R_\mathfrak{u}$ of R such that we have $\mathfrak{u} = \bigoplus_{\alpha \in R_\mathfrak{u}} V_\alpha$. The set $R_\mathfrak{u}$ is non-empty. We fix a root $\alpha_1 \in R_\mathfrak{u}$. Let R_1 be the irreducible root system that is a direct factor of R and that contains α_1. Let G_1 be the semisimple, normal subgroup of G that corresponds naturally to R_1. So G_1 is almost simple, is normalized by T, and moreover we have $V_{\alpha_1} \subset \mathrm{Lie}(G_1) = \mathrm{Lie}(G_1 \cap T) \bigoplus_{\alpha \in R_1} V_\alpha$. Let $G_{\pm\alpha_1}$ be the unique reductive subgroup of G that contains T and whose Lie algebra is $\mathrm{Lie}(T) \oplus V_{\alpha_1} \oplus V_{-\alpha_1}$, cf. [2, Vol. III, Exp. XXII, 5.4.7 and 5.10.1]. Let $S_{\pm\alpha_1}$ be the derived group of $G_{\pm\alpha_1}$; it is a subgroup of G_1. Moreover $S_{\pm\alpha_1}$ is isomorphic to either $\mathbf{PGL}_2$ or $\mathbf{SL}_2$ and we have $V_{\alpha_1} \oplus V_{-\alpha_1} \subset \mathrm{Lie}(S_{\pm\alpha_1})$.

If G_1 is adjoint, then it is a direct factor of G. So as $V_{\alpha_1} \subset \mathrm{Lie}(G_1) \cap \mathfrak{u}$ and as $\mathrm{Lie}(G_1) \cap \mathfrak{u}$ is a G_1-module, to prove the Lemma we can assume $G = G_1$; thus $R = R_1$. As $\mathfrak{u}$ is stable under the normalizer of T in G, the set $R_\mathfrak{u}$ is stable under the action of the Weyl group of the root system R. Any two roots of R of equal length, are conjugate under the Weyl group of R (cf. [1, Ch. IV, §3, Prop. 11]). So we also have $-\alpha_1 \in R_\mathfrak{u}$. As $\mathfrak{u}$ is a G-module, it is an ideal of $\mathrm{Lie}(G)$ and so also a Lie subalgebra of $\mathrm{Lie}(G)$. Thus we have $[V_{\alpha_1}, V_{-\alpha_1}] \subset \mathfrak{u} \cap \mathrm{Lie}(T) = 0$. So $[V_{\alpha_1}, V_{-\alpha_1}] = 0$. This implies that $\mathrm{Lie}(S_{\pm\alpha_1})$ is not isomorphic to $\mathrm{Lie}(\mathbf{SL}_2)$. But for $p \neq 2$, the central isogeny $\mathbf{SL}_2 \to \mathbf{PGL}_2$ is étale and thus we can identify $\mathrm{Lie}(\mathbf{SL}_2)$ with $\mathrm{Lie}(\mathbf{PGL}_2)$. So we must have $p = 2$ and $S_{\pm\alpha_1}$ must be isomorphic to $\mathbf{PGL}_2$. But it is well known that $S_{\pm\alpha_1}$ is isomorphic to $\mathbf{PGL}_2$ if and only if G is isomorphic to $\mathbf{SO}_{2n+1}$ for some $n \geqslant 1$ and α_1 is a short root. [Argument: see [4, 3.8]; strictly speaking the arguments of loc. cit. are stated over a finite field but they also apply over any separably closed field.] We conclude that G is isomorphic to $\mathbf{SO}_{2n+1}$ for some $n \geqslant 1$ and that any root $\alpha_1 \in R_\mathfrak{u}$ is short; thus $R_\mathfrak{u}$ is the set of short roots of R. □

2.1. An isogeny

Let $p = 2$. Let $\{e_0, \ldots, e_{2n}\}$ be the standard k-basis of $V := k^{2n+1}$. Let $V_0 := ke_0$. We take G to be the semisimple subgroup of $\mathbf{GL}_V$ that is the $\mathbf{SO}$-group of the quadratic form $Q(x) := x_0^2 + x_1x_{n+1} + \cdots + x_nx_{2n}$ on V; here $x := (x_0, \ldots, x_{2n})$. Let $\{e_{i,j} \mid 0 \leqslant i, j \leqslant 2n\}$ be the standard k-basis of $\mathrm{End}_k(V)$. The formula $\Psi(x, y) := Q(x + y) - Q(x) - Q(y)$ defines an alternating form on V that is fixed by G and whose kernel is V_0; so Ψ induces a non-degenerate alternating form Ψ_0 on $W_0 := V/V_0$. Let N_n be the kernel of the natural isogeny $G \to \mathbf{Sp}(W_0, \Psi_0)$. For a commutative k-algebra A, we have $N_n(A) = \{1_{V \otimes_k A} + \sum_{i=1}^{2n} \beta_i e_{0,i} \mid \beta_i \in A,\ \beta_i^2 = 0\}$. Thus $N_n \tilde{\to} \boldsymbol{\alpha}_2^{2n}$. Moreover $\mathrm{Lie}(N_n) = \bigoplus_{i=1}^{2n} ke_{0,i}$ is the ideal $\mathfrak{u}$ of the previous paragraph.

Theorem 2.2. *Let U be a normal, unipotent subgroup scheme of G. We have*:

(a) *if U is nontrivial, then $p = 2$ and $G_{k^{\mathrm{sep}}}$ has a direct factor that is isomorphic to $\mathbf{SO}_{2n+1}$ for some $n \geqslant 1$;*
(b) *if U is nontrivial and if G is an absolutely simple, adjoint group over k, then U is the unique nontrivial, normal, unipotent subgroup scheme of G;*
(c) *there exists a maximal normal, unipotent subgroup scheme $U_{\max}$ of G which is compatible with field extensions*;
(d) *if U is nontrivial, there exists $l \geqslant 1$ such that $U \tilde{\to} \boldsymbol{\alpha}_2^{2l}$.*

2.2. Proof of Theorem 2.2

Let $\mathfrak{u} := \mathrm{Lie}(U)$. As G is reductive, U is connected and finite. We prove (a). So as U is also a nontrivial group scheme, $\mathfrak{u}$ is a nonzero G-submodule of $\mathrm{Lie}(G)$. As U has a trivial intersection with any maximal torus T of G, we have $\mathfrak{u} \cap \mathrm{Lie}(T) = 0$. So (a) holds, cf. Lemma 2.1.

To prove (b), we can assume G is $\mathbf{SO}_{2n+1}$ (cf. (a)). Let $N_n \lhd G$ be as in Subsection 2.1. We have $\mathrm{Lie}(N_n) = \mathfrak{u}$, cf. end of Subsection 2.1. Thus each $\boldsymbol{\alpha}_2$ copy of $N_n \tilde{\to} \boldsymbol{\alpha}_2^{2n}$ is a subgroup scheme of U. So $N_n \lhd U$. Thus $U/N_n \lhd$

G/N_n. But G/N_n is isomorphic to $\mathbf{Sp}_{2n}$, cf. Subsection 2.1. So G/N_n has no nontrivial, normal, unipotent subgroup scheme, cf. (a). Thus U/N_n is the trivial group scheme. So $U = N_n$ is uniquely determined. So (b) holds.

To prove (c) and (d), we recall that $G_{k^{\mathrm{sep}}}$ is split (cf. [2, Vol. III, Exp. XXII, 2.4]). As $G_{k^{\mathrm{sep}}}$ is split, we have a direct product decomposition

$$G_{k^{\mathrm{sep}}} = G_0 \times_{k^{\mathrm{sep}}} G_1 \times_{k^{\mathrm{sep}}} \cdots \times_{k^{\mathrm{sep}}} G_s$$

such that the group G_0 has no direct factor that is isomorphic to an $\mathbf{SO}_{2n+1}$ group and each group G_i with $i \in \{1, \ldots, s\}$ is isomorphic to $\mathbf{SO}_{2n_i+1}$ for some $n_i \geqslant 1$. For $i \in \{0, \ldots, s\}$, let $\pi_i : G_{k^{\mathrm{sep}}} \twoheadrightarrow G_i$ be the natural projection. Let $U_i := U_{k^{\mathrm{sep}}}/(U_{k^{\mathrm{sep}}} \cap \mathrm{Ker}(\pi_i))$; we have $U_i \tilde{\to} \mathrm{Im}(U_{k^{\mathrm{sep}}} \to G_i)$. The group scheme U_0 is trivial, cf. (a). For $i \in \{1, \ldots, s\}$, let $U_i^{\max}$ be the unique nontrivial, normal, unipotent subgroup scheme of G_i (cf. Subsection 2.1 and (b)); so U_i is either trivial or $U_i^{\max}$. From this and the fact that $U \lhd G$, we get that $U_{k^{\mathrm{sep}}} = \prod_{i=1}^{s} (U_{k^{\mathrm{sep}}} \cap G_i) \tilde{\to} \prod_{i=1}^{s} U_i$. We also get that $G_{k^{\mathrm{sep}}}$ has a unique maximal normal, unipotent subgroup scheme $U_{k^{\mathrm{sep}}}^{\max} := \prod_{i=1}^{s} U_i^{\max}$.

Using standard Galois descent, we get (from the uniqueness part) that $U_{k^{\mathrm{sep}}}^{\max}$ is the extension to k^{sep} of the maximal normal, unipotent subgroup scheme $U^{\max}$ of G. For any algebraically closed field K that contains k, $U_K^{\max}$ is the maximal normal, unipotent subgroup scheme of G_K. So (c) holds.

For $i \in \{1, \ldots, s\}$ the intersection $U_{k^{\mathrm{sep}}} \cap G_i \tilde{\to} U_i$ is either trivial or $N_{n_i} \tilde{\to} \boldsymbol{\alpha}_2^{2n_i}$; thus $U_{k^{\mathrm{sep}}} \tilde{\to} \boldsymbol{\alpha}_2^{2l}$, where $l := \sum_{i \in E} n_i \geqslant 1$ for some non-empty subset E of $\{1, \ldots, s\}$. The group scheme of automorphisms of $\boldsymbol{\alpha}_2^{2l}$ is a $\mathbf{GL}_{2l}$ group. As $H^1(k, \mathbf{GL}_{2l})$ is the trivial set, each form of $\boldsymbol{\alpha}_2^{2l}$ is trivial. So we have an isomorphism $U \tilde{\to} \boldsymbol{\alpha}_2^{2l}$ over k. So (d) holds.

3. Proof of Theorems 1.1 and 1.2

In 3.1 we show that (for a given pair (k, G)) the four statements S_1 to S_4 are equivalent. In Subsection 3.2 we combine Subsection 3.1 and Section 2 to prove Theorems 1.1 and 1.2.

3.1. The equivalence of the four statements

We show that S_1 is equivalent to S_3. We use the notations of S_3. Let $U_0 := \mathrm{Ker}(\widetilde{G} \to \mathbf{GL}_{\mathrm{gr}(V)})$; it is a normal, unipotent subgroup scheme of $\widetilde{G}$. As the $\widetilde{G}$-module $\mathrm{gr}(V)$ is semisimple, U_0 is the maximal normal, unipotent subgroup scheme of $\widetilde{G}$. Thus U_0/U is the maximal normal, unipotent subgroup scheme of $G = \widetilde{G}/U$. As $\mathrm{Ker}(\tilde{\rho}_{\mathrm{ss}}) = U_0/U$, we get that $\tilde{\rho}_{\mathrm{ss}}$ is faithful if and only if G has no nontrivial, normal, unipotent subgroup scheme. Thus S_1 is equivalent to S_3.

By taking $\widetilde{G} = G$, we get that S_3 implies S_2. We check that S_2 implies S_4. Let $f : G \to G'$, $T \subset G$, $T' \subset G'$, and $f_T : T \to T'$ be as in S_4. Let $U := \mathrm{Ker}(f)$. Let U^0 be the identity component of U. The normal, finite, étale subgroup U/U^0 of G/U^0 is contained in the center of G/U^0 and so also in $\mathrm{Im}(T \to G/U^0) \tilde{\to} T$. As the intersection $U \cap T$ is trivial, we get $U = U^0$. A similar argument shows that U has no subgroup scheme that is of multiplicative type and nontrivial. Thus U is unipotent, cf. [2, Vol. II, Exp. XVII, 4.3.1]. The normal, unipotent subgroup scheme U of G is contained in the kernel of any semisimple representation of G. So if S_2 holds, then U is trivial; thus S_4 holds.

We check that S_4 implies S_1. Let U be a normal, unipotent subgroup scheme of G. Let $f : G \twoheadrightarrow G/U =: G'$ be the natural epimorphism. Let $T' := \mathrm{Im}(T \to G')$. As the intersection $U \cap T$ is trivial, $f_T : T \to T'$ is an embedding. So if S_4 holds, then f is a closed embedding and thus U is trivial. So S_4 implies S_1.

3.2. End of the argument

It suffices to prove Theorem 1.1 for statement S_1, cf. Subsection 3.1. So Theorem 1.1 follows from Theorem 2.2 (a). Let now $p = 2$. To prove Theorem 1.2, we can assume $G_{k^{\mathrm{sep}}}$ has a direct factor that is isomorphic to $\mathbf{SO}_{2n+1}$ for some $n \geqslant 1$ (cf. Theorem 2.2 (a)). To check that G has a nontrivial, normal, unipotent subgroup scheme U, we can assume (cf. Theorem 2.2 (c)) that $k = k^{\mathrm{sep}}$; so G has a direct factor that is isomorphic to $\mathbf{SO}_{2n+1}$ and thus U exists (cf. Subsection 2.1). This ends the proofs of Theorems 1.1 and 1.2.

Acknowledgements

We would like to thank J.-P. Serre for many comments and much encouragement to write this note. We would like to thank the University of Arizona for good conditions in which to write this Note.

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