Abstract

We prove that the Cauchy problem associated with a Pfaff system with coefficients in $L^p_{\text{loc}}, p > 2$, in a connected and simply-connected open subset $\Omega$ of $\mathbb{R}^2$ has a unique solution provided that its coefficients satisfies a compatibility condition in the distributional sense. To cite this article: S. Mardare, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

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Résumé


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admet une solution unique $Y \in C^2(\Omega; \mathbb{M}^{q \times \ell})$ si les coefficients $A_i$ appartiennent à l’espace $C^1(\Omega; \mathbb{M}^\ell)$ et satisfont la condition de compatibilité

$$\partial_1 A_2 + A_1 A_2 = \partial_2 A_1 + A_2 A_1 \quad \text{dans } \Omega.$$  

L’objet de cette Note est d’établir que ce résultat reste vrai sous les hypothèses affaiblies que les coefficients $A_i$ appartiennent à l’espace $L^p_{\text{loc}}(\Omega; \mathbb{M}^\ell)$, $p > 2$, la condition de compatibilité ci-dessus étant alors satisfaite au sens des distributions (voir Théorème 3.2 dans la version anglaise). La preuve repose sur deux résultats principaux : un résultat de stabilité pour les systèmes de Pfaff à coefficients dans $L^p(\Omega)$ établi dans le Théorème 2.1 et un résultat d’approximation, sous la contrainte non linéaire (1), des champs de matrices $A_i$ établi dans le Lemme 3.1 de la version anglaise.

La démonstration complète de ces résultats, esquissée dans la version anglaise, se trouve dans [5].

1. Preliminaries

The notations $\mathbb{M}^{q \times \ell}$, $\mathbb{M}^\ell$, $\mathcal{S}^\ell$ et $\mathcal{S}_2^\ell$ respectively designate the set of all matrices with $q$ rows and $\ell$ columns, the set of all square matrices of order $\ell$, the set of all symmetric matrices of order $\ell$, and the set of all positive definite symmetric matrices of order $\ell$. For vectors $v = (v_i)$ and matrices $A = (A_{ij})$, we define the norms

$$\|v\| = \sum_i |v_i| \quad \text{and} \quad \|A\| := \sum_{i,j} |A_{ij}|.$$  

A generic point in $\mathbb{R}^2$ is denoted $x = (x_1, x_2)$ and partial derivatives of first and second order are denoted $\partial_1 = \frac{\partial}{\partial x_1}$ and $\partial_2 = \frac{\partial^2}{\partial x_2 \partial x_1}$. An open ball with radius $R$ centered at $x \in \mathbb{R}^2$ is denoted $B_R(x)$, or $B_R$ if its center is irrelevant in the subsequent analysis.

The space of distributions over an open set $\Omega \subset \mathbb{R}^2$ is denoted $\mathcal{D}'(\Omega)$. The usual Sobolev spaces being denoted $W^{m,p}(\Omega)$, we let

$$W^{m,p}_{\text{loc}}(\Omega) := \{ f \in \mathcal{D}'(\Omega); \ f \in W^{m,p}(U) \text{ for all open set } U \subset \Omega \},$$

where the notation $U \subset \Omega$ means that the closure of $U$ in $\mathbb{R}^2$ is a compact subset of $\Omega$. The closure in $W^{1,p}(\Omega)$ of the space of all indefinitely derivable functions with compact support in $\Omega$ is denoted $W^{1,p}_{0}(\Omega)$. If $p > 2$, the classes of functions in $W^{1,p}(\Omega)$ are identified with their continuous representatives, as in the Sobolev imbedding theorem (see, e.g., Adams [1]). For matrix-valued and vector-valued function spaces, we shall use the notations $W^{m,p}(\Omega; \mathbb{M}^{q \times \ell})$, $W^{m,p}(\Omega; \mathbb{R}^\ell)$, etc.

The Lebesgue spaces $L^p(\Omega; \mathbb{R}^\ell)$ and $L^p(\Omega; \mathbb{M}^{q \times \ell})$ are equipped with the norms

$$\|v\|_{L^p(\Omega)} = \sum_i \|v_i\|_{L^p(\Omega)} \quad \text{and} \quad \|A\|_{L^p(\Omega)} = \sum_{i,j} \|A_{ij}\|_{L^p(\Omega)},$$

and the Sobolev spaces $W^{1,p}(\Omega; \mathbb{M}^{q \times \ell})$ and $W^{2,p}(\Omega; \mathbb{M}^{q \times \ell})$ are equipped with the norms

$$\|Y\|_{W^{1,p}(\Omega)} = \|Y\|_{L^p(\Omega)} + \sum_i \|\partial_i Y\|_{L^p(\Omega)} \quad \text{and} \quad \|Y\|_{W^{2,p}(\Omega)} = \|Y\|_{W^{1,p}(\Omega)} + \sum_{i,j} \|\partial_{ij} Y\|_{L^p(\Omega)}.$$

The following theorem gathers the Morrey and Sobolev inequalities with explicit constants that will be used in the next sections:

**Theorem 1.1.** Let $B_R \subset \mathbb{R}^2$ be an open ball of radius $R > 0$ and let $p > 2$. Then there exists constants $C_1$, $C_2 > 0$ depending only on $p$ such that
Theorem 2.1. Let \( \Omega \) be a connected open subset of \( \mathbb{R}^2 \), let \( x^0 \in \Omega \), let \( p > 2 \), let \( A^2_n \in L^p(\Omega; M^2) \) and \( Y^m \in W^{1,p}(\Omega; M^{2 \times 2}) \) be sequences of matrix fields that satisfy the Pfaff systems

\[
\partial_i Y^m = Y^m A^n_i \quad \text{in} \quad \mathcal{D}'(\Omega; M^{2 \times 2}), \quad n \in \mathbb{N},
\]

and assume that there exists a constant \( M \) such that \( \sum_i \|A^i_n\|_{L^p(\Omega)} + \|Y^m(x^0)\| \leq M \) for all \( n \in \mathbb{N} \). Then, for each open set \( K \Subset \Omega \), there exist a constant \( C > 0 \) such that

\[
\|Y^m - Y^m\|_{W^{1,p}(K)} \leq C \left( \sum_i \|A^i_n - A^i_m\|_{L^p(\Omega)} + \|Y^m(x^0) - Y^m(x^0)\| \right) \quad \text{for all} \quad n, m \in \mathbb{N}.
\]

**Proof.** Fix any open ball \( B_R = B_R(x) \Subset \Omega \), where \( x \in K \) and \( 2C_1 MR^{1-1/p} < 1 \) (\( C_1 \) is the constant appearing in Theorem 1.1). Using Morrey’s inequality (see Theorem 1.1), viz

\[
\|Y^m - Y^m\|_{L^p(B_R)} \leq \left( \|Y^m - Y^m\|(x) \right) + C_1 R^{1-2/p} \sum_i \|\partial_i(Y^m - Y^m)\|_{L^p(B_R)},
\]

and the relation \( \partial_i(Y^m - Y^m) = (Y^m - Y^m) A^i_n + Y^m (A^i_n - A^i_m) \), we obtain on the one hand that

\[
\sum_i \|\partial_i(Y^m - Y^m)\|_{L^p(B_R)} \leq 2M \|Y^m - Y^m\|(x) + 2 \|Y^m\|_{L^\infty(B_R)} \sum_i \|A^i_n - A^i_m\|_{L^p(B_R)}.
\]

Using again Morrey’s inequality together with relations \( \partial_i Y^m = Y^m A^n_i \) and \( 2C_1 M R^{1-1/p} < 1 \), we deduce on the other hand that

\[
\|Y^m\|_{L^\infty(B_R)} \leq \|Y^m\|(x) + C_1 R^{1-2/p} \sum_i \|\partial_i Y^m\|_{L^p(B_R)} \leq 2 \|Y^m\|(x),
\]

and, by joining the point \( x \) to \( x^0 \) by a broken line formed by \( N \) segments of length \( < R \), that

\[
\|Y^m\|_{L^\infty(B_R)} \leq 2^N \|Y^m(x^0)\| \leq 2^N M,
\]

where the number \( N \) depends only on \( x_0, \Omega \) and \( K \). Using this inequality in inequality (3) gives

\[
\sum_i \|\partial_i(Y^m - Y^m)\|_{L^p(B_R)} \leq 2M \|Y^m - Y^m\|(x) + 2^{N+1} M \sum_i \|A^i_n - A^i_m\|_{L^p(B_R)}.
\]

Then we infer from inequalities (2) and (4) that there exists a constant \( C > 0 \) such that

\[
\|Y^m - Y^m\|_{W^{1,p}(B_R(x))} \leq C \left( \|Y^m - Y^m\|(x) + \sum_i \|A^i_n - A^i_m\|_{L^p(\Omega)} \right).
\]

Let now the open set \( K \Subset \Omega \) be covered with a finite number of balls of radius \( R \). By joining the center \( x \) of any such ball to the given point \( x^0 \) with a broken line formed by \( N \) segments of length \( < R \) (the number \( N \) depends
only on $x_0$, $\Omega$ and $K$), we show by a recursion argument that there exists another constant $C$ independent of $n, m$ such that
\[
\|Y^n - Y^m\|_{W^{1,p}(B_R(x))} \leq C \left(\|Y^n - Y^m\|(x_0)^0 \right) + \sum_i \|\bar{A}^n_i - \bar{A}^m_i\|_{L^p(\Omega)}.
\]
Since this inequality is valid for any ball $B_R(x)$ in the chosen covering of $K$, summing all such inequalities gives the announced inequality. □

An immediate consequence of Theorem 2.1 is the following uniqueness result:

**Corollary 2.2.** Let $\Omega$ be an connected open subset of $\mathbb{R}^2$, let $p > 2$, and let there be given matrix fields $A_i \in L^p_{loc}(\Omega; \mathbb{M}^\ell)$ and $Y, \bar{Y} \in W^{1,p}_{loc}(\Omega; \mathbb{M}^{q\times \ell})$ that satisfy the relations
\[
\partial_i Y = Y A_i \quad \text{and} \quad \partial_i \bar{Y} = \bar{Y} A_i \quad \text{in} \ D'(\Omega; \mathbb{M}^{q\times \ell}).
\]
Assume that there exists a point $x^0 \in \Omega$ such that $Y(x^0) = \bar{Y}(x^0)$. Then $Y(x) = \bar{Y}(x)$ for all $x \in \Omega$.

### 3. Existence of the solution to Pfaff systems with $L^p$ coefficients in dimension two

Let $\Omega \subset \mathbb{R}^2$ be a connected and simply-connected open set and let there be given a point $x_0 \in \Omega$ and a matrix $Y_0 \in \mathbb{M}^{q\times \ell}$. Then it is well-known (see, e.g., Thomas [7]), that the (Cauchy problem associated with the) Pfaff system
\[
\partial_i Y = Y A_i \quad \text{in} \ \Omega, \ i \in \{1, 2\},
\]
\[
Y(x^0) = Y^0,
\]
has a unique solution if the coefficients $A_i$ belong to the space $C^1(\Omega; \mathbb{M}^\ell)$ and satisfy the compatibility condition
\[
\partial_1 A_2 + A_1 A_2 = \partial_2 A_1 + A_2 A_1 \quad \text{in} \ \Omega.
\]
This result has been subsequently improved by Hartman and Wintner [3] (under the assumption that $A_i \in C^0(\Omega; \mathbb{M}^\ell)$) and by Mardare [4] (under the assumption that $A_i \in L^p_{loc}(\Omega; \mathbb{M}^\ell)$). Our objective is to establish an existence and uniqueness result under the assumption that $A_i \in L^p_{loc}(\Omega; \mathbb{M}^\ell)$, $p > 2$. The key ingredient in establishing this result is the following lemma.

**Lemma 3.1.** Let $\Omega$ be an open subset of $\mathbb{R}^2$, let $p > 2$, and let matrix fields $A_i \in L^p(\Omega; \mathbb{M}^\ell)$ be given that satisfy the relations
\[
\partial_1 A_2 + A_1 A_2 = \partial_2 A_1 + A_2 A_1 \quad \text{in} \ D'(\Omega; \mathbb{M}^\ell).
\]
Then, for each open ball $B_R \subset \Omega$ whose radius satisfies
\[
R < \min \left(1, \left\{C(p) \left(\|A_1\|_{L^p(\Omega)} + \|A_2\|_{L^p(\Omega)}\right)\right\}^{p/(2-p)}\right),
\]
where $C(p)$ is a constant depending only on $p$, there exist sequences of matrix fields $\bar{A}^n_i \in C^\infty(\overline{B_R}; \mathbb{M}^\ell)$, $n \in \mathbb{N}$, that satisfy the relations
\[
\partial_1 \bar{A}^n_2 + \bar{A}^n_1 A_2 = \partial_2 \bar{A}^n_1 + A_2 \bar{A}^n_1 \quad \text{in} \ B_R,
\]
\[
\bar{A}^n_i \to A_i \quad \text{in} \ L^p(B_R; \mathbb{M}^\ell) \ \text{as} \ n \to \infty.
\]
Proof. The key of the proof is the following change of unknowns:

\[ A_1 = \partial_1 U - \partial_2 V \quad \text{and} \quad A_2 = \partial_2 U + \partial_1 V \quad \text{in} \quad B_R, \]

where \( U \in W^{1,p}(B_R; \mathbb{M}^\ell) \) and \( V \in W^{2,p/2}_0(B_R; \mathbb{M}^\ell) := W^{2,p/2}(B_R, \mathbb{M}^\ell) \cap W_0^{1,p}(B_R, \mathbb{M}^\ell) \). This system has a solution that can be computed as follows: First, define \( V \in W^{2,p/2}_0(B_R; \mathbb{M}^\ell) \) as the solution to the Poisson equation

\[ \Delta V = A_2 A_1 - A_1 A_2 \quad \text{in} \quad D'(B_R, \mathbb{M}^\ell), \]

then define \( U \in W^{1,p}(B_R; \mathbb{M}^\ell) \) as a solution to the Poincaré system (assumption (5) is used here)

\[ \partial_1 U = A_1 + \partial_2 V \quad \text{and} \quad \partial_2 U = A_2 - \partial_1 V. \]

Now, the approximating sequences for the fields \( A_i \) are defined in the following way: First, the field \( U \) is approximated with the smooth matrix fields \( U^n \in C^\infty(\overline{B}_R; \mathbb{M}^\ell) \) defined by taking the convolution of (an extension to \( \mathbb{R}^2 \)) of \( U \) with a sequence of mollifiers, so that

\[ U^n \to U \quad \text{in} \quad W^{1,p}(B_R; \mathbb{M}^\ell) \quad \text{as} \quad n \to \infty. \]

Then the field \( V_n, \ n \in \mathbb{N}, \) is defined as the solution to the system

\[ \Delta V_n = (\partial_2 U^n + \partial_1 V^n)(\partial_1 U^n - \partial_2 V^n) - (\partial_1 U^n - \partial_2 V^n)(\partial_2 U^n + \partial_1 V^n) \quad \text{in} \quad D'(\Omega; \mathbb{M}^\ell), \]

\[ V^n = 0 \quad \text{on} \quad \partial \Omega. \]

We prove that this nonlinear system has at least one solution of class \( C^\infty \) in \( \overline{B}_R \) by using the implicit function theorem (see, e.g., Schwartz [6]) applied to the mapping

\[ f : W^{1,p}(B_R; \mathbb{M}^\ell) \times W^{2,p/2}_0(B_R; \mathbb{M}^\ell) \to L^{p/2}(B_R; \mathbb{M}^\ell), \]

defined by \( f(X, Y) = \Delta Y - (\partial_2 X + \partial_1 Y)(\partial_1 X - \partial_2 Y) + (\partial_1 X - \partial_2 Y)(\partial_2 X + \partial_1 Y) \). To this end, we show that the mapping \( f(U, V) \) is an isomorphism from the space \( W^2_0(B_R; \mathbb{M}^\ell) \) to the space \( L^{p/2}(B_R; \mathbb{M}^\ell) \) by using the Lax–Milgram lemma, Theorem 1.1, and assumption (6) on the size of the ball \( B_R \). Consequently, there exist open subsets \( O_1 \subset W^{1,p}(B_R; \mathbb{M}^\ell) \) and \( O_2 \subset W^{2,p/2}_0(B_R; \mathbb{M}^\ell) \) and a mapping \( \psi \in C^1(O_1; O_2) \) such that \( U \in O_1, V \in O_2 \) and \( (X, Y) \in O_1 \times O_2 ; \ f(X, Y) = 0 \) \( \Rightarrow \) \( (X, \psi(X)) \in O_1 \). In particular, for \( X = U^n \) there exists \( V^n := \psi(U^n) \) such that \( f(U^n, V^n) = 0 \). The regularity properties of second order elliptic partial differential equations (see, e.g., Gilbarg and Trudinger [2]) show that in fact \( V^n \in C^\infty(\overline{B}_R) \). Moreover, since \( \psi \) is continuous and since \( U^n \to U \) in \( W^{1,p}(B_R; \mathbb{M}^\ell) \), it follows that \( V^n \to V \) in \( W^{2,p/2}(B_R; \mathbb{M}^\ell) \), hence in \( W^{1,p}(B_R; \mathbb{M}^\ell) \) by the Sobolev imbedding theorem (see, e.g., Adams [1]).

Finally, we define the fields \( A_1^n := \partial_1 U^n - \partial_2 V^n \) and \( A_2^n := \partial_2 U^n + \partial_1 V^n \), and prove that they satisfy the required conditions of the lemma. \( \square \)

We are now in a position to prove the main result of this Note.

**Theorem 3.2.** Let \( \Omega \) be a connected and simply connected open subset of \( \mathbb{R}^2 \), let \( x^0 \in \Omega \), let \( p > 2 \), let \( Y^0 \in M^{q \times \ell} \), and let matrix fields \( A_i \in L^{\infty}_0(\Omega; M^{q \times \ell}) \) be given that satisfy the relations

\[ \partial_1 A_2 + A_1 A_2 = \partial_2 A_1 + A_2 A_1 \quad \text{in} \quad D'(\Omega; M^{q \times \ell}). \]

Then the Pfaff system

\[ \partial_i Y = Y A_i \quad \text{in} \quad D'(\Omega; M^{q \times \ell}), \]

\[ Y(x^0) = Y^0 \quad \text{in} \quad \Omega \]

has one and only one solution \( Y \in W^{1,p}_0(\Omega; M^{q \times \ell}). \)
Proof. We first prove the following local existence result: For each open ball $B_r := B_r(x^0) \subseteq \Omega$ whose radius satisfies relation (6) of the previous lemma, there exists a field $Y \in W^{1,p}(B_r; \mathbb{M}^{q \times \ell})$ that satisfies the Pfaff system

$$\forall i \in \{1, 2\}, \quad \partial_i Y = Y A_i \quad \text{in} \quad \mathcal{D}'(B_r; \mathbb{M}^{q \times \ell}),$$

$$Y(x^0) = Y^0. \quad (8)$$

We find this solution as the limit of a sequence of solutions to some Pfaff systems with smooth coefficients. For, fix an open ball $B_R \subseteq \Omega$ such that $B_r \subseteq B_R$. Then Lemma 3.1 shows that there exist sequences of matrix fields $A^n_1, A^n_2 \in C^\infty(B_R; \mathbb{M}^{\ell})$ that satisfy

$$\partial_1 A^n_2 + A^n_1 A^n_2 = \partial_2 A^n_1 + A^n_2 A^n_1 \quad \text{in} \quad B_R,$$

$$A^n_1 \to A_1 \quad \text{and} \quad A^n_2 \to A_2 \quad \text{in} \quad L^p(B_R; \mathbb{M}^{\ell}) \quad \text{as} \quad n \to \infty.$$

Since the coefficients $A^n_1$ and $A^n_2$ are smooth, the classical result on Pfaff systems (see, e.g., Thomas [7]) shows that there exists a matrix field $Y^n \in C^\infty(B_R; \mathbb{M}^{q \times \ell})$ that satisfies

$$\forall i \in \{1, 2\}, \quad \partial_i Y^n = Y^n A^n_i \quad \text{in} \quad B_R,$$

$$Y^n(x^0) = Y^0. \quad (9)$$

By the stability result of Theorem 2.1, there exists a constant $C > 0$ such that

$$\|Y^n - Y^m\|_{W^{1,p}(B_r)} \leq C \sum_{i=1}^2 \|A^n_i - A^m_i\|_{L^p(B_R)} \quad \text{for all} \quad m, n \in \mathbb{N},$$

which means that $(Y^n)$ is a Cauchy sequence in the space $W^{1,p}(B_r; \mathbb{M}^{q \times \ell})$. Since this space is complete, there exists a field $Y \in W^{1,p}(B_r; \mathbb{M}^{q \times \ell})$ such that $Y^n \to Y$ in $W^{1,p}(B_r; \mathbb{M}^{q \times \ell})$ as $n \to \infty$. In addition, the Sobolev imbedding $W^{1,p}(B_r; \mathbb{M}^{q \times \ell}) \subset C^0(B_r; \mathbb{M}^{q \times \ell})$ shows that $Y^n(x^0) \to Y(x^0)$ in $\mathbb{M}^{q \times \ell}$ as $n \to \infty$. Then we deduce that the field $Y$ satisfies the Pfaff system (8) by passing to the limit as $n \to \infty$ in the equations of system (9).

Now, we define a global solution to the Pfaff system (7) as in the proof of Theorem 3.1 of [4], by gluing together some sequences of local solutions along curves starting from the given point $x^0$. We prove that this definition is unambiguous thanks to the uniqueness result of Corollary 2.2 and to the simply-connectedness of the set $\Omega$.

That this solution is unique follows from Corollary 2.2. □

Remark 1. The assumption that $p > 2$ of the theorem is optimal since in order to properly define $Y(x^0)$, the space $W^{1,p}(\Omega)$ (to which the components of the matrix field $Y$ belong) should be imbedded in the space of continuous functions.

References