Algebra

The structure of certain rigid tensor categories

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Received 8 December 2003; accepted after revision 29 March 2005

Available online 22 April 2005

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Abstract

We consider rigid tensor categories over a field of characteristic zero in which some exterior power of each object is zero. To cite this article: P. O’Sullivan, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

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Résumé


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Théorème 0.1 (cf. [1], 16.1.1 et 13.7.1). Soit $A$ une catégorie $k$-tensorielle positive avec $\text{End}_A(1) = k$. Alors la projection $A \rightarrow A/R(A)$ admet un quasi-inverse à droite. Si $T$ est un tel quasi-inverse à droite et si $D$ est une catégorie $k$-tensorielle positive semi-simple avec $\text{End}_D(1) = k$, alors tout foncteur $k$-tensoriel $D \rightarrow A$ se factorise, à isomorphisme tensoriel près, à travers $T$.

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doi:10.1016/j.crma.2005.03.018
Nous prouvons en fait le Lemme 0.2 ci-dessous. Le Théorème 0.1 dit de plus que deux quasi-inverses à droite quelconques $T_1$ et $T_2$ de la projection coïncident à isomorphisme tensoriel près, mais ceci se déduit du Lemme 0.2 en prenant $D = A/\mathcal{R}(A) \otimes_k A/\mathcal{R}(A)$ et pour $D \to A$ le foncteur défini par $T_1$ et $T_2$.

**Lemme 0.2.** Si $A$ et $D$ sont comme dans le Théorème 0.1, alors tout foncteur $k$-tensoriel $D \to A$ se factorise, à isomorphisme tensoriel près, à travers un quasi-inverse à droite à la projection $A \to A/\mathcal{R}(A)$.

Pour prouver le Lemme 0.2 on peut supposer, par les deux lemmes suivants, que $A = \text{Mod}(A)$ et que $D \to A$ est $A \otimes -$ pour $A$ une algèbre (commutative) dans la catégorie $\text{Ind}(D)$ des ind-objets de $D$, où $\text{Mod}(A)$ est l’enveloppe pseudo-abélienne de la catégorie de $A$-modules libres $A \otimes M$ avec $M \in \text{Ob} D$.

**Lemme 0.3.** Soit $(M_i)_{i \in I}$ une famille d’objets dans une catégorie $k$-tensorielle positive $A$, avec $\dim M_i = n_i \in \mathbb{N}$. Notons $V_i$ la représentation standard du facteur $\text{GL}(n_i)$ de $\prod_{i \in I} \text{GL}(n_i)$. Alors il existe un foncteur $k$-tensoriel $H : \text{Rep}_k(\prod_{i \in I} \text{GL}(n_i)) \to A$ avec $HV_i = M_i$ pour $i \in I$.

**Lemme 0.4.** Soient $H : C \to A$ un foncteur $k$-tensoriel. Supposons que $C$ soit rigide et que $\text{Hom}_A(H-, 1)$ sur $C$ soit ind-représentable. Alors $H$ est, à isomorphisme tensoriel près, le composé d’un foncteur $k$-tensoriel $A \otimes - : C \to \text{Mod}(A)$ avec un foncteur $k$-tensoriel pleinement fidèle $\text{Mod}(A) \to A$.

Pour prouver le Lemme 0.2 lorsque $D \to A$ est $A \otimes -$ : $D \to \text{Mod}(A)$, soit $\hat{A}$ un quotient simple de $A$. Par le Lemme 0.5 ci-dessous on peut identifier $A/\mathcal{R}(A)$ et $\text{Mod}(\hat{A})$ de sorte que $A \to \hat{A}/\mathcal{R}(A)$ soit défini par la projection $A \to \hat{A}$. Le Lemme 0.2 résulte donc du Lemme 0.6.

**Lemme 0.5.** Soient $D$ une catégorie $k$-tensorielle positive semi-simple avec $\text{End}_D(1) = k$ et $A$ une algèbre dans $\text{Ind}(D)$ avec $\text{Hom}_{\text{Ind}(D)}(1, A) = k$. Alors $\text{Mod}(A)$ est semi-simple si et seulement si $A$ est simple.

**Lemme 0.6.** Si $D$ et $A$ sont comme dans le Lemme 0.5, alors la projection de $A$ sur un quotient simple admet un inverse à droite dans la catégorie des algèbres dans $\text{Ind}(D)$.

Soit $A$ une catégorie $k$-tensorielle positive. Par les Lemmes 0.3 et 0.4, il existe un couple $(G, X)$, avec $G$ un $k$-groupe pro-réductif et $X$ un $G$-schéma affine, tel que $A$ soit $\otimes$-équivalente à la catégorie des fibrés vectoriels $G$-équivariants sur $X$. Si $\text{End}_A(1) = k$ et si $k$ est algébriquement clos, on peut déduire du Théorème 0.1 qu’il existe un tel $(G, X)$ unique à isomorphisme près tel que $X$ ait un $k$-point fixé par $G$.

1. **Introduction**

Fix a field $k$ of characteristic 0. By a $k$-tensor category we mean a $k$-linear, pseudo-Abelian, symmetric monoidal category. The exterior power $\wedge^r M$ of an object $M$ in a $k$-tensor category $A$ is defined as the image of the anti-symmetriser in $\text{End}_A(M^{\otimes r})$. We say that $M$ is positive if $M$ has a dual $M^\vee$ and $\wedge M = 0$ for some $r$, and that $A$ is positive if it is essentially small with every object positive. A $k$-tensor category is said to be rigid if every object has a dual. Let $A$ be a rigid $k$-tensor category with $\text{End}_A(1) = k$. Then $A$ has a unique maximal tensor ideal $\mathcal{R}(A)$, consisting of the $f : M \to N$ such that $\text{tr}(fg) = 0$ for each $g : N \to M$. We have $\mathcal{R}(A) = 0$ if and only if each non-zero $M \to 1$ in $A$ is a retraction.

A category will be called semisimple if it is Abelian with every object projective. By [3], Théorème 7.1, a $k$-tensor category is semisimple positive with $\text{End}(1) = k$ if and only if it is semisimple Tannakian. Theorem 1.1 is proved in Section 6, and in Section 7 it is indicated how structure theorems can be deduced from it. A totally different proof of Theorem 1.1 has been given by André and Kahn in [1].
Theorem 1.1 (cf. [1], 16.1.1 and 13.7.1). Let $\mathcal{A}$ be a positive $k$-tensor category with $\text{End}_\mathcal{A}(1) = k$. Then $\mathcal{A}/\mathcal{R}(\mathcal{A})$ is semisimple and positive, and the projection $\mathcal{A} \rightarrow \mathcal{A}/\mathcal{R}(\mathcal{A})$ has a right quasi-inverse. If $T$ is such a right quasi-inverse and $D$ is a semisimple positive $k$-tensor category with $\text{End}_D(1) = k$, then any $k$-tensor functor $D \rightarrow \mathcal{A}$ factors up to tensor isomorphism through $T$.

2. Algebras and modules in a tensor category

Let $\mathcal{A}$ be a $k$-tensor category. If $A$ is an algebra (always understood to be commutative) in $\mathcal{A}$, we denote by $\text{MOD}(A)$ the category of $A$-modules. When $A$ is Abelian and its tensor product preserves colimits, $\text{MOD}(A)$ is an Abelian $k$-tensor category and the forgetful functor $\text{MOD}(A) \rightarrow \mathcal{A}$ creates limits and colimits. We say that the algebra $A$ is simple if it is simple as an object of $\text{MOD}(A)$.

Suppose that $\mathcal{A}$ is the category $\text{Ind}(C)$ of ind-objects ([2], I 8) of a $k$-tensor category $\mathcal{C}$. Then we denote by $\text{Mod}(A)$ the pseudo-Abelian hull of the full subcategory of $\text{MOD}(A)$ of free $A$-modules $A \otimes M$ on objects $M$ in $\mathcal{C}$. It is a $k$-tensor category. We have a $k$-tensor functor $A \otimes \cdot : \mathcal{C} \rightarrow \text{Mod}(A)$, and also $B \otimes A : \text{Mod}(A) \rightarrow \text{Mod}(B)$ defined by a given $A \rightarrow B$. If $\mathcal{C}$ is positive then $\text{Mod}(A)$ is positive.

Let $G$ be an affine $k$-group. We denote by $\text{Rep}_k(G)$ the $k$-tensor category of finite-dimensional representations of $G$, and by $\text{REP}_k(G)$ the $k$-tensor category of all representations. We have $\text{Rep}_k(G) = \text{Ind}(\text{Rep}_k(G))$. An algebra $A$ in $\text{REP}_k(G)$ is the same as a $G$-algebra, and we write $\text{MOD}(G, A)$ for $\text{MOD}(A)$ and $\text{Mod}(G, A)$ for $\text{Mod}(A)$. The $k$-group $G$ is proreductive if and only if $\text{Rep}_k(G)$ (or equivalently $\text{Rep}_k(G)$) is semisimple.

3. Positive objects and representations of the general linear group

Lemma 3.1. The dimension $\dim M = \text{tr} 1_M \in \text{End}_\mathcal{A}(1)$ of a positive object $M$ in a $k$-tensor category $\mathcal{A}$ can be written as $\sum_{j=1}^s n_j e_j$, where $n_j \in \mathbb{N}$ and the $e_j$ are mutually orthogonal idempotents of $\text{End}_\mathcal{A}(1)$ with $\sum_{j=1}^s e_j = 1$. If $e_j \neq 0$ for all $j$, then $\max_{1 \leq j \leq r} n_j$ is the least $m \in \mathbb{N}$ such that $\bigwedge^{m+1} M = 0$.

Proof. (For the case where $\text{End}_\mathcal{A}(1) = k$ see also [1], 9.1.7.) We may suppose $\text{End}_\mathcal{A}(1) \neq 0$. Write $\alpha_r \in \text{End}_\mathcal{A}(M^{\otimes r})$ for the antisymmetrizer, and $u : 1 \rightarrow M \otimes M' \otimes \cdot$ and $c : M \otimes M' \otimes M'' \rightarrow 1$ for the canonical morphisms. If $\dim M = d$, the "contraction" $(M^{\otimes r} \otimes c) \circ (\alpha_{r+1} \otimes M') \circ (M^{\otimes r} \otimes u)$ of $\alpha_{r+1}$ is $(d-r)/(r+1)\alpha_r$. Suppose that $\alpha_{m+1} = 0$ and $\alpha_m \neq 0$. Then inductively we have $(\alpha_{m+1}^d) = \text{tr} \alpha_{m+1} = 0$, whence the first statement, with $n_j \leq m$ when $e_j \neq 0$. Also $(d-m)\alpha_m = 0$, whence the second statement. \qed

Lemma 3.2. Let $(M_i)_{i \in I}$ be a family of positive objects in a $k$-tensor category $\mathcal{A}$, with $\dim M_i = m_i \in \mathbb{N}$. Denote by $V_i$ the standard representation of the factor GL$(m_i)$ of $\prod_{i \in I} \text{GL}(m_i)$. Then there is a $k$-tensor functor $H : \text{Rep}_k(\prod_{i \in I} \text{GL}(m_i)) \rightarrow \mathcal{A}$ with $H V_i = M_i$ for $i \in I$.

Proof. For simplicity we give the proof in the case where $I$ has one element, and omit the indices $i$. The general case is similar, with the free rigid $k$-tensor category on a family $(N_i)_{i \in I}$ replacing $\mathcal{F}$ below. Let $G_r$ be the symmetric group of degree $r$, and write $\alpha_r \in k[G_r]$ for the antisymmetrizer and $\sigma_r : k[G_r] \rightarrow \text{End}(L^{\otimes r})$ for the canonical homomorphism associated to an object $L$. Let $\mathcal{F}$ be the free rigid $k$-tensor category on one object $N$ (see [4], 1.26). We have $\text{End}[\mathcal{F}(1) = k[N]$ with $t = \dim N$, the $\sigma_N^r$ induce isomorphisms $k[N][G_r] \cong \text{End}[\mathcal{F}(N^{\otimes r})$, and $\text{Hom}[\mathcal{F}(N^{\otimes r}, V^{\otimes s}) = 0$ for $r \neq s$. If $\mathcal{J}$ is the tensor ideal of $\mathcal{F}$ generated by $t - m$ and $\sigma_N^{m+1} a_{m+1}$, let $\mathcal{F}^\prime$ be the pseudo-Abelian hull of $\mathcal{F}/\mathcal{J}$, and $\overline{N}$ the image of $N$ in $\mathcal{F}^\prime$. Since $\dim V = \dim M = m$ and $\sigma_N^{m+1} a_{m+1} = 0$ and (by Lemma 3.1) $\sigma_N^{m+1} a_{m+1} = 0$, there are by the universal property of $\mathcal{F}$ $k$-tensor functors $R : \mathcal{F} \rightarrow \text{Rep}_k(\text{GL}(m))$ and $S : \mathcal{F}^\prime \rightarrow \mathcal{A}$ with $R \overline{N} = V$ and $S \overline{N} = M$. Now $\sigma_N^r$ is surjective, with kernel $0$ for $r \leq m$ and generated by $a_{m+1} \in k[G_{m+1}] \subset k[G_r]$. For $r > m$, $R$ is an equivalence by rigidity of $\mathcal{F}$. Now take $H = S \circ R'$ with $R'$ quasi-inverse to $R$. \qed
Lemma 3.3. Let $H : C \to A$ be a $k$-tensor functor. Suppose that $C$ is rigid and that $\text{Hom}_A(H-,1)$ on $C$ is ind-representable ([2], I 8.2.2). Then $H$ is tensor isomorphic to the composite of a $k$-tensor functor $A \otimes - : C \to \text{Mod}(A)$ with a fully faithful $k$-tensor functor $\text{Mod}(A) \to A$.

Proof. Write $\widetilde{H} = \text{Ind}(H)$ and let $f : \widetilde{H} A \to 1$ define an isomorphism $\psi : \text{Hom}_{\text{Ind}(C)}( -, A) \xrightarrow{\sim} \text{Hom}_{\text{Ind}(A)}(\widetilde{H}, 1)$. There is a unique algebra structure on $A$ such that $f$ is a morphism of algebras. Then $H$ is tensor isomorphic to $F \circ (A \otimes -)$ with $F = 1 \otimes_{\widetilde{H} A} \widetilde{H} : \text{Mod}(A) \to A$ defined by $f$. The factorisation $\text{Hom}_A(A \otimes M, A) \xrightarrow{\text{tr}} \text{Hom}_{\text{Ind}(C)}(M, A) \xrightarrow{\psi} \text{Hom}_A(HM, 1) \xrightarrow{\sim} \text{Hom}_A(F(A \otimes M), FA)$ of $F_A \otimes_M A$ for $M \in C$, together with the rigidity of $\text{Mod}(A)$, shows that $F$ is fully faithful. □

Lemma 3.4. A $k$-tensor category $A$ is positive if and only if there exists a proreductive $k$-group $G$ and a $G$-algebra $A$ such that $A$ is a $k$-tensor equivalent to $\text{Mod}(G, A)$.

Proof. Suppose that $A$ is positive. If $M$ is an object of $A$, and if $n_j$ and $e_j$ are as in Lemma 3.1, then since the image $N_j$ of $e_j : 1 \to 1$ has dimension $e_j$, the object $M \oplus \bigoplus_n N_j^{m-n_j}$ has dimension $m \in \mathbb{N}$ for $m \geq \max n_j$. Hence by Lemma 3.2 there is a $G = \prod_{i \in I} \text{GL}(m_i)$ and a $k$-tensor functor $H : \text{Rep}_k(G) \to A$ such that every object of $A$ is a direct summand of one in the image of $H$. Since $\text{Rep}_k(G)$ is semisimple, $\text{Hom}_A(H-, 1)$ is exact and hence ind-representable ([2], I 8.3.3). Thus Lemma 3.3 gives a $k$-tensor equivalence $\text{Mod}(G, A) \to A$. The converse is immediate. □

4. Simple algebras and semisimplicity

Lemma 4.1 (see [1] 8.2.4). Let $A$ be a positive $k$-tensor category with $\text{End}_A(1) = k$. Then $A/\mathcal{R}(A)$ is semisimple and positive with finite-dimensional hom $k$-spaces. If $A$ is semisimple then $\mathcal{R}(A) = 0$.

Lemma 4.2. Let $\mathcal{D}$ be a semisimple positive $k$-tensor category with $\text{End}_\mathcal{A}(1) = k$ and let $A$ be an algebra in $\text{Ind}(\mathcal{D})$ with $\text{Hom}_{\text{Ind}(\mathcal{D})}(1, A) = k$. Then $\text{Mod}(A)$ is semisimple if and only if $A$ is simple. When this is so $\text{MOD}(A)$ is semisimple, and $\text{MOD}(A) = \text{Ind}(\text{Mod}(A))$.

Proof. The isomorphisms $\text{Hom}_A(A \otimes N, -) \simeq \text{Hom}_{\text{Ind}(\mathcal{D})}(N, -)$ show that $\text{Hom}_A(L, -)$ is exact and preserves filtered colimits when $L \in \text{Mod}(A)$. Now $A$ is simple if and only if each non-zero $M \to A$ in $\text{MOD}(A)$ is an epimorphism, and by Lemma 4.1 $\text{Mod}(A)$ is semisimple if and only if each non-zero $M \to A$ in $\text{Mod}(A)$ is a retraction. Since $\text{Hom}_A(A, -)$ is exact, the first statement follows. Suppose now that $\text{Mod}(A)$ is semisimple. Any $A$-module $L$ is the cokernel of an $A$-morphism $l : A \otimes K' \to A \otimes K$, with for example $K = L$. Writing $K$ and $K'$ as filtered colimits of objects in $\mathcal{D}$ and appropriately reindexing, we may express $l$ as a filtered colimit of $A$-morphisms $l_j : A \otimes K'_j \to A \otimes K_j$ with $K'_j, K_j \in \mathcal{D}$. Then $L$ is the filtered colimit of the cokernels of the $l_j$, which by semisimplicity lie in $\text{Mod}(A)$. Thus $\text{MOD}(A) = \text{Ind}(\text{Mod}(A))$. The objects of $\text{Mod}(A)$ are of finite length by Lemma 4.1, so each object of $\text{MOD}(A)$ is a coproduct of simple objects of $\text{Mod}(A)$, whence $\text{MOD}(A)$ is semisimple. □

5. Algebras with action of a proreductive group

Lemma 5.1 (Magid [7], Theorem 4.5). Let $G$ be an affine $k$-group of finite type and let $A$ be a $G$-algebra with $A^G = k$. Then $A$ is a simple $G$-algebra if and only if $\text{Spec}(A)$ is homogeneous under $G$.

Lemma 5.2. Let $G$ be a reductive $k$-group, let $A$ be a finitely generated $G$-algebra with $A^G = k$, and let $J \neq A$ be a $G$-ideal of $A$. Then the canonical homomorphism $A \to \text{lim}_n A/J^n$, where the limit is taken in $\text{REP}_k(G)$, is an isomorphism.
Proof. It suffices to show that \( \text{Hom}_G(V, A) \rightarrow \lim_n \text{Hom}_G(V, A/J^n) \) is bijective for \( V \in \text{Rep}_k(G) \). The surjectivity is clear since \( \text{Hom}_G(V, A) \) is finite-dimensional over \( k = A^G \) (e.g. [8], Theorem 3.25) and \( \text{Hom}_G(V, -) \) is exact. To prove the injectivity, we may by extending the scalars assume \( k \) algebraically closed. It suffices to check that then \( \bigcap_n J^n = 0 \), or equivalently (e.g. [9], Chapter IV, Theorem 12') that if \( p \) is an associated prime of \( 0 \subset A \) then \( J + p \neq A \). Since \( k \) is algebraically closed, a \( (k) \)-subspace of a representation of \( G \) is a \( G \)-subspace, as can be seen by reducing to the finite-dimensional case. Thus \( p_0 = \bigcap_{g \in G(k)} gp \) is a \( G \)-ideal of \( A \). It follows that \( A \rightarrow A/J \times A/p_0 \) is not surjective since \( \dim_k(A/J \times A/p_0)^G > 1 \), so \( J + p_0 \neq A \). Since each \( gp \) lies in the finite set of associated primes of \( 0, \) some \( J + gp \neq A \), so \( J + p = g^{-1}(J + gp) \neq A. \) \( \square \)

Lemma 5.3 (cf. [6], Corollaire 2). Let \( G \) be a proreductive \( k \)-group and \( A \) be a \( G \)-algebra with \( A^G = k. \) Then \( A \) has a unique simple \( G \)-quotient \( \tilde{A}. \) If \( D \) is a simple \( G \)-subalgebra of \( A, \) then the projection \( A \rightarrow \tilde{A} \) has a right inverse in the category of \( G \)-algebras over \( D. \)

Proof. By Zorn's Lemma \( A \) has a maximal \( G \)-ideal \( J. \) If \( J 
\neq = \mathbb{A} \) is a \( G \)-ideal of \( A, \) then \( A \rightarrow A/J \times A/J' \) is not surjective since \( \dim_k(A/J \times A/J')^G > 1, \) so \( J + J' \neq A \) and \( J \subset J' \). Thus \( \tilde{A} = A/J \times A \) exists and is unique. The second statement is proved in (1), (2) and (3) below. We note that if \( G \) is of finite type then by Lemma 5.1 \( \tilde{A} \) and \( D \) are finitely generated \( k \)-algebras and \( \tilde{A} \) is smooth over \( D. \)

(1) Suppose that \( G \) is of finite type and that \( J^2 = 0. \) Write \( E \) for the set of \( k \)-algebra homomorphisms \( \tilde{A} \rightarrow A \) over \( D \) right inverse to \( A \rightarrow \tilde{A}, \) and let \( V \subset \tilde{A} \) be a finite-dimensional \( G \)-subspace which contains \( k \) and generates \( \tilde{A}. \) We may regard \( E \) as a subset of \( \text{Hom}_k(V, A) = V^∨ \otimes_k A. \) Now \( E \neq \emptyset \) by smoothness of \( \tilde{A} \) over \( D, \) and if \( e \in E \) then \( e - e \) is the \( k \)-space of derivations of \( A \) over \( D \) with values in \( J. \) Thus the \( k \)-subspace \( \tilde{E} \) of \( V^∨ \otimes_k A \) generated by \( E \) is a \( G \)-subspace, and the evaluation \( V^∨ \otimes_k A \rightarrow A \) at \( 1 \in \tilde{E} \) defines a surjective \( G \)-homomorphism \( \tilde{E} \rightarrow k \subset A \) with fibre \( E \) above 1. Since \( G \) is reductive, the set \( \tilde{E}^G \cap E \) of \( \text{Hom}_k(G) \)-algebras over \( D \) right inverse to \( A \rightarrow \tilde{A} \) is non-empty.

(2) Suppose that \( G \) is of finite type. Replacing \( A \) with its \( G \)-subalgebra generated by \( D \) and a lifting to \( A \) of a finite set of generators of \( \tilde{A}, \) we may assume that \( \tilde{A} \) is finitely generated. Then \( A = \lim_n A/J^n \in \text{Rep}(G) \) by Lemma 5.2. Thus it is enough to show that a morphism \( \tilde{A} \rightarrow A/J^n \) of \( G \)-algebras over \( D \) can be lifted to \( \tilde{A} \rightarrow A/J^{n+1}. \) In fact the pullback of \( A/J^{n+1} \rightarrow A/J^n \) along \( \tilde{A} \rightarrow A/J^n \) has kernel of square 0, and so has a right inverse over \( D \) by (1).

(3) Consider the general case. By Zorn's Lemma, \( A \) has a maximal simple \( G \)-subalgebra \( C \) containing \( D. \) It suffices to show that \( C \rightarrow \tilde{A} \) is an isomorphism. To do this we show that \( \text{Hom}_C(C, H) \rightarrow \text{Hom}_C(C, H) \) is an isomorphism for each normal \( k \)-subgroup \( H \) of \( G \) with \( G_1 = G/H \) of finite type. If \( B \) is a simple \( G \)-algebra then \( B^H \) is a simple \( G_1 \)-algebra, because \( I = (B1)^H \) for a \( G_1 \)-ideal \( I \) of \( B^H. \) Thus \( \text{MOD}(G_1, C^H) \) is semisimple by Lemma 4.2, so every \( (G_1, C^H) \)-module is a direct summand of a free \( (G_1, C^H) \)-module \( C^H \otimes_k M, \) with \( M \in \text{REP}_k(G_1). \) By the isomorphisms \( \text{Hom}_{G_1}(M, C^H \otimes_k N) \cong \text{Hom}_M(C \otimes_k N) \) for \( M, N \in \text{REP}_k(G_1), \) the \( k \)-tensor functor \( F = C \otimes_{C^H} \rightarrow \text{MOD}(G_1, C^H) \rightarrow \text{MOD}(G, C) \) is thus fully faithful. Further by semisimplicity of \( \text{MOD}(G, C), \) every \( (G, C) \)-submodule of one of the essential image of \( F \) is in the essential image of \( F. \) Thus \( FC_1 \) is a simple \( G \)-algebra if \( C_1 \) is a simple \( G_1 \)-algebra over \( C^H, \) and \( FC_1 \neq C \) if \( C_1 \neq C^H. \) Since an embedding \( C_1 \rightarrow A^H \subset A \) over \( C^H \) defines an embedding \( FC_1 \rightarrow A \) over \( C, \) it follows that \( C^H \) is a maximal simple \( G_1 \)-subalgebra of \( A^H. \) Applying (2) with \( G_1, A^H, \tilde{A}^H \) and \( C^H \) for \( G, A, \tilde{A} \) and \( D \) then shows that \( C^H \rightarrow \tilde{A}^H \) is an isomorphism. \( \square \)

6. Proof of Theorem 1.1

In the category of \( k \)-tensor categories and \( k \)-tensor isomorphism classes of \( k \)-tensor functors, the coproduct of \( C_1 \) and \( C_2 \) is the pseudo-Abelian hull \( C_1 \otimes_k C_2 \) of the category with objects \( \text{Ob} C_1 \times \text{Ob} C_1, \) hom \( k \)-spaces tensor products of those of \( C_1 \) and \( C_2, \) and a suitable tensor structure. If \( C_1 \) and \( C_2 \) are semisimple positive with \( \text{End}(I) = k, \) so is \( C_1 \otimes_k C_2; \) its endomorphism algebras are semisimple by Lemma 4.1, and it is generated by objects in the image of the \( C_1 \rightarrow C_1 \otimes_k C_2, \) and so is positive by Lemmas 3.1 and 3.2.
Lemma 6.1. Let $D$ and $A$ be as in Lemma 4.2. Then $A$ has a unique simple quotient $\hat{A}$, and the projection $A \to \hat{A}$ has a right inverse in the category of algebras in $\text{Ind}(D)$.

Proof. By Lemma 3.4 we may assume $D = \text{Mod}(G, D)$ with $G$ reductive and $D^G = k$. Then $D$ is $G$-simple and $\text{Ind}(D) = \text{MOD}(G, D)$, by Lemma 4.2. The result thus follows from Lemma 5.3. □

Lemma 6.2. If $A$ and $D$ are as in Theorem 1.1, then every $k$-tensor functor $D \to A$ factors, up to tensor isomorphism, through a right quasi-inverse to the projection $A \to A/\mathcal{R}(A)$.

Proof. By Lemmas 3.1 and 3.2 there is an essentially surjective $k$-tensor functor $\text{Rep}_k(G) \to A$, with $G$ a product of general linear groups. Since $\mathcal{D} \to A$ factors up to tensor isomorphism through $\mathcal{D} \otimes_k \text{Rep}_k(G) \to A$, we may by replacing $D$ with $\mathcal{D} \otimes_k \text{Rep}_k(G)$ assume that $\mathcal{D} \to A$ is essentially surjective. Applying Lemma 3.3, we may then suppose that $A = \text{Mod}(A)$ for an algebra $A$ in $\text{Ind}(\mathcal{D})$, and that $\mathcal{D} \to A$ is $A \otimes -$.

By Lemma 6.1 $A$ has a simple quotient $\hat{A}$, and by Lemma 4.2 $\text{Mod}(\hat{A})$ is semisimple. Since $\hat{A} \otimes_A - : \text{Mod}(A) \to \text{Mod}(\hat{A})$ is full, it factors through a $k$-tensor equivalence $\text{Mod}(A)/\mathcal{R}(\text{Mod}(A)) \to \text{Mod}(\hat{A})$, by Lemma 4.1. It now suffices to note that if $\hat{A} \to A$ is right inverse to $A \to \hat{A}$ as in Lemma 6.1, then $A \otimes_{\hat{A}} - : \text{Mod}(\hat{A}) \to \text{Mod}(A)$ is right quasi-inverse to $\hat{A} \otimes_A -$, and $A \otimes - : \mathcal{D} \to \text{Mod}(A)$ factors up to tensor isomorphism as $(A \otimes_{\hat{A}} -) \circ (\hat{A} \otimes -)$. □

To prove Theorem 1.1, it remains after Lemmas 4.1 and 6.2 only to show that any two right quasi-inverses $T_1$ and $T_2$ to the projection $P : A \to A/\mathcal{R}(A)$ are tensor isomorphic. If $\simeq$ denotes tensor isomorphism, then $T_i \simeq T_i^\prime T_i^\prime$, with $T_i^\prime : A/\mathcal{R}(A) \otimes_k A/\mathcal{R}(A) \to A$. By Lemma 6.2, $T^\prime \simeq TS$ for some $T$ with $PT \simeq \text{Id}$. Then $S \simeq PT^\prime$, so $T_i \simeq TST_i^\prime \simeq TPT_i^\prime \simeq T$ and $T_1 \simeq T_2$.

7. Structure theorems

If $G$ is a reductive $k$-group and $A$ is a $G$-algebra, then $\text{Mod}(G, A)$ is $k$-tensor equivalent to the category $\text{Vec}(G, X)$ of $G$-equivariant vector bundles over $X = \text{Spec}(A)$, since each such bundle is a direct summand of the pullback along $X \to \text{Spec}(k)$ of a representation of $G$. Thus by Lemma 3.4, any positive $k$-tensor category is $k$-tensor equivalent to $\text{Vec}(G, X)$ for some reductive $G$ and affine $G$-scheme $X$.

Let $\hat{A}$ be a positive $k$-tensor category with $\text{End}(\hat{A}) = k$, and suppose that $k$ is algebraically closed. It can be shown by applying Lemma 3.3 to the right quasi-inverse of Theorem 1.1 that, among pairs $(G, X)$ with $G$ reductive, $X$ affine and such that $\text{Vec}(G, X)$ is $k$-tensor equivalent to a $\hat{A}$, there is one $(G_0, X_0)$ for which $X_0$ has a $k$-point fixed by $G_0$. Further for any other such $(G, X)$ there is an embedding $G_0 \to G$ of $k$-groups such that $X$ is $G$-isomorphic to the homogeneous fibre space $G \times_{G_0} X_0$.

References