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C. R. Acad. Sci. Paris, Ser. I 340 (2005) 587–592
COMPTES RENDUS MATHEMATIQUE
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Dynamical Systems

Alien limit cycles near a Hamiltonian 2-saddle cycle

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Received 10 January 2005; accepted after revision 12 March 2005

Available online 22 April 2005

Presented by Étienne Ghys

Abstract

It is known that perturbations from a Hamiltonian 2-saddle cycle Γ can produce limit cycles that are not covered by the Abelian integral, even when it is generic. These limit cycles are called alien limit cycles. This phenomenon cannot appear in the case that Γ is a periodic orbit, a non-degenerate singularity, or a saddle loop. In this Note, we present a way to study this phenomenon in a particular unfolding of a Hamiltonian 2-saddle cycle, keeping one connection unbroken at the bifurcation. ***To cite this article: M. Caubergh et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Cycles limites étrangers près d'un 2-polycycle hamiltonien. Il est connu que les perturbations d'un 2-polycycle hamiltonien Γ peuvent produire des cycles limites qui ne sont pas reliés aux zéros de l'intégrale abélienne associée, même si elle est générique. Ces cycles limites sont appelés cycles limites étrangers. Ce phénomène ne peut pas apparaitre dans le cas où Γ est une orbite périodique, une singularité non-dégénérée, ou bien un laçet homocline. Dans cette Note, nous présentons une méthode pour étudier ce phénomène dans le cas d'un déploiement particulier de 2-polycycle Hamiltonien, préservant l'une des deux connections pendant la bifurcation. ***Pour citer cet article : M. Caubergh et al., C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

On considère des familles lisses $(X_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$, de champs de vecteurs du plan, telles que pour $\varepsilon = 0$, $X_{(\nu,0)} = X_H$ est hamiltonien et que le portrait de phase de $X_{(\nu,0)}$ contienne un 2-polycycle Γ dans la frontière d'un continuum d'orbites périodiques. Un tel 2-polycycle consiste en deux points de selle hyperboliques, s_1 et s_2 connectés par

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1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.03.009

deux orbites hétéroclines. On rappelle qu'un cycle limite est une orbite périodique isolée. Le nombre maximum de cycles limites de $X_{(\nu,\varepsilon)}$ bifurquant de Γ pour de petites perturbations du paramètre (ν,ε) près de $(\nu_0,0)$, est appelé cyclicité de Γ. Traditionnellement, pour étudier les cycles limites de $X_{(\nu,\varepsilon)}$ près de Γ, on associe à cette famille de champs de vecteurs une famille d'applications de déplacement $(\delta_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$, définie comme la différence entre l'application de retour de Poincaré sur une section transverse et l'identité. Les zéros isolés de l'application de déplacement $\delta_{(\nu,\varepsilon)}$ correspondent aux cycles limites du champ de vecteurs $X_{(\nu,\varepsilon)}$. Des informations importantes sont données par la partie linéaire $I \equiv \frac{\partial}{\partial\varepsilon}\delta_{|\varepsilon=0}$, qui peut être calculée comme une intégrale et est appelée *l'intégrale abélienne* associée.

Lorsque Γ est une orbite périodique, un point elliptique non-dégénéré ou bien un laçet homocline, il est bien connu que d'une façon générique, l'étude de l'intégrale Abelienne I est suffisante pour obtenir la connaissance complète du nombre des cycles limites et de leurs bifurcations [5,3]. D'un autre côté, il est montré dans [2] que cela n'est plus le cas lorsque Γ est un p-polycycle ($p \geqslant 2$) et que le déploiement casse au moins l'une des connections au cours de la bifurcation. Si Γ est un 2-polycycle, avec l'une et seulement l'une des connections cassée, il est montré dans [2], que le déploiement peut produire génériquement quatre cycles limites, alors que l'intégrale abelienne associée produit au plus trois zéros. En conséquence, au moins l'un des cycles limites n'est pas associé à un zéro de l'intégrale abélienne ; un tel cycle limite est appelé cycle limite étranger.

Le but de cette Note est de construire pour chaque $k \geqslant 2$, une famille déployant un 2-polycycle en maintenant l'une des connections tout en produisant au moins $k-2$ cycles limites étrangers. On suppose qu'il existe des coordonnées locales au voisinage de chacun des points de selle s_1 et s_2 dans lesquelles le champ $X_{(\nu,\varepsilon)}$, est linéaire pour toute valeur du paramètre, avec un même rapport d'hyperbolicité en s_1 et s_2. On dira que le déploiement est *rigide*. Sous cette condition précisée par (1), l'application de déplacement réduite $\bar{\Delta}$ prend la forme (2). Le développement asymptotique de cette application dépend de deux fonctions lisses f_1 et f_2. Les jets infinis de ces fonctions sont exprimés dans (3). On suppose que les coefficients de ces jets, qui sont des fonctions lisses du paramètre (ν,ε), vérifient les conditions génériques (i), (ii) et (iii) introduites ci-dessous. Nous pouvons alors prouver le théorème suivant :

Théorème 0.1. *Si $(X_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$ est un déploiement rigide de codimension $2k-1$ (la codimension est définie ci-dessous), alors au moins $k-2$ cycles limites étrangers bifurquent dans cette famille.*

1. Introduction

We consider smooth families of planar vector fields $(X_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$, such that for $\varepsilon=0$, $X_{(\nu,0)} = X_H$ is Hamiltonian and the phase portrait of $X_{(\nu,0)}$ contains a 2-saddle cycle Γ in the boundary of a continuum of periodic orbits. A 2-saddle cycle consists of two hyperbolic saddles, s_1 and s_2, connected by two heteroclinic orbits. Recall that a limit cycle is an isolated periodic orbit. The maximal number of limit cycles of $X_{(\nu,\varepsilon)}$ that can arise from Γ, after small perturbations of the parameter (ν,ε) near $(\nu_0,0)$, is called the cyclicity at Γ. Traditionally to study limit cycles of $X_{(\nu,\varepsilon)}$ near Γ, a family of displacement maps $(\delta_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$, as difference of the Poincaré map and the identity along a transverse segment, is associated to the family $(X_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$ such that isolated zeroes of $\delta_{(\nu,\varepsilon)}$ correspond to limit cycles of $X_{(\nu,\varepsilon)}$. Important information is given by the linearisation $I \equiv \frac{\partial}{\partial\varepsilon}\delta_{|\varepsilon=0}$, that can be computed as an integral, called the related Abelian integral.

In case Γ is a periodic orbit, a non-degenerate elliptic point or a saddle loop [5,3], it is well-known that generically the study of the Abelian integral I suffices to get full knowledge on the number of limit cycles and their bifurcations. In [2], it is established that this is no longer the case when Γ is a p-saddle cycle ($p \geqslant 2$), if the unfolding breaks at least one connection at the bifurcation. If Γ is a 2-saddle cycle, of which exactly one connection gets broken, it is proven in [2], that the unfolding can produce generically 4 limit cycles, while the related Abelian integral can produce at most 3 zeroes. Hence, there is at least one limit cycle that is not covered by a zero of the related Abelian integral; such a limit cycle is called an alien limit cycle. We present for each $k \geqslant 2$, a not too degenerate family, unfolding a 2-saddle cycle and keeping a connection unbroken, with at least k alien limit cycles.

This Note is organised as follows. First, for each k, the specific family and its degeneracy conditions are described. Next, the principal result is stated, and a sketch of its proof is provided.

We suppose (see [4]) that the vector fields $X_{(\nu,\varepsilon)}$ and $-X_{(\nu,\varepsilon)}$ at the saddle points s_1 and s_2 respectively, are linear with as respective expression:

$$\begin{cases} \dot y = -y(1+\varepsilon\alpha), \\ \dot x = x \end{cases} \quad \text{and} \quad \begin{cases} \dot w = -w(1+\varepsilon\alpha), \\ \dot z = z. \end{cases} \tag{1}$$

The unbroken connection is represented by respectively $\{x=0\}$ and $\{z=0\}$; α is a non-zero constant, equal in both expressions. Condition (1) will be used throughout this note, and we will shortly say that the unfolding $(X_\lambda)_\lambda$ is *rigid*. In the study of limit cycles near a 2-saddle cycle, it is convenient to replace the traditional displacement map δ by a difference map $\varepsilon\bar\Delta$, as in [2], defined on $\{y=1\}$ in terms of x in the first expression of (1) and expressing, in the coordinate w, the difference between the backward and the forward transition towards $\{z=1\}$ in the second normal form expression of (1). In particular, isolated zeroes of $\bar\Delta(\cdot,\nu,\varepsilon)$, for $\varepsilon>0$, correspond to limit cycles of $X_{(\nu,\varepsilon)}$, and the Abelian integral satisfies $I_\nu \equiv \bar\Delta_{|\varepsilon=0}(\cdot,\nu)$. As in [2], we can write

$$\bar\Delta(x,\nu,\varepsilon) = \beta(\nu,\varepsilon) - u(\nu,\varepsilon)x^{1+\varepsilon\alpha} + x^{1+\varepsilon\alpha}\big[f_2(x,\nu,\varepsilon) - \big(1+\varepsilon u(\nu,\varepsilon)\big)f_1(x^{1+\varepsilon\alpha},\nu,\varepsilon)\big], \tag{2}$$

where β, u, f_1 and f_2 are C^∞ functions, in their respective variables, that are related to regular transitions along the connections (i.e. not passing through the saddle points).

It is well-known that the Abelian integral $I(x,\nu) = \bar\Delta(x,\nu,0)$ admits an asymptotic expansion in a logarithmic scale, in which we only encounter linear terms in $\log x$, while for $n>0$, the terms $\frac{\partial^n}{\partial\varepsilon^n}\bar\Delta(x,\nu,0)$ admit an asymptotic expansion in an enlarged logarithmic scale [2]; more precisely, besides linear terms in $\log x$ also non-linear terms in $\log x$ show up in their expansion. For the particular family we deal with, even the linear terms in $\log x$ disappear from the expansion of I. More precisely, if

$$j_\infty\big(f_1(\cdot,\nu,\varepsilon)\big)_0(y) = \sum_{i=1}^{\infty}\eta_i^{(1)}(\nu,\varepsilon)y^i \quad \text{and} \quad j_\infty\big(f_2(\cdot,\nu,\varepsilon)\big)_0(x) = \sum_{i=1}^{\infty}\eta_i^{(2)}(\nu,\varepsilon)x^i, \tag{3}$$

then I has the asymptotic expansion

$$j_\infty(I_\nu)_0(x) = j_\infty\big(\bar\Delta(\cdot,\nu,0)\big)_0(x) = \beta(\nu,0) - u(\nu,0)x + \sum_{i=2}^{\infty}\big(\eta_{i-1}^{(2)}(\nu,0) - \eta_{i-1}^{(1)}(\nu,0)\big)x^i.$$

Before formulating some genericity conditions on (2), we introduce the notation

$$\eta_{1\dots j}^{(i)}(\nu,\varepsilon) = \big(\eta_1^{(i)}(\nu,\varepsilon),\dots,\eta_j^{(i)}(\nu,\varepsilon)\big), \quad \forall i=1,2,\ \forall j \geqslant 1.$$

Now we say that the rigid unfolding $(X_\lambda)_\lambda$ is generic of codimension $2k-1$ ($k \geqslant 2$) at $\lambda = 0$ (i.e. $\nu=\nu_0=0, \varepsilon=0$) if the following *genericity conditions* are satisfied:

(i) $\beta(0) = u(0) = (\eta_i^{(2)} - \eta_i^{(1)})(0) = 0$, $\forall 1 \leqslant i \leqslant k-2$,
(ii) $(\eta_{k-1}^{(2)} - \eta_{k-1}^{(1)})(0) \neq 0$ and $\eta_{k-1}^{(1)}(0) \neq 0$,
(iii) $Q : (\mathbb{R}^{2k-1},0) \to (\mathbb{R}^{2k-1},0) : (\nu,\varepsilon) \mapsto (\beta, u, \eta_{1\dots k-2}^{(1)}(\nu,\varepsilon), \eta_{1\dots k-2}^{(2)}(\nu,\varepsilon), \varepsilon)$ is a local diffeomorphism at 0.

We hence assume that $\nu \in \mathbb{R}^{2k-2}$ and without loss of generality we will introduce $(\beta, u, \eta_{1\dots k-2}^{(1)}, \eta_{1\dots k-2}^{(2)})$ as new independent parameters that we still denote by ν. It follows from these genericity conditions that exactly k zeroes are produced by the Abelian integral. We however expect that under these genericity conditions the cyclicity in the rigid unfolding $(X_\lambda)_\lambda$ is $2k-1$, inducing the existence of $k-1$ alien limit cycles. We can prove the following slightly weaker result:

Theorem 1.1. *If $(X_{(\nu,\varepsilon)})_{(\nu,\varepsilon)}$ is a rigid unfolding of codimension $2k-1$ (as described above), then at least $k-2$ alien limit cycles bifurcate in this family.*

2. Sketch of the proof of Theorem 1.1

Since by Rolle's theorem, the Abelian integral I_ν can have at most k zeroes, we merely have to prove that $\bar{\Delta}$ has at least $2k-2$ isolated zeroes for $\varepsilon > 0$ small. To obtain this result, we first expand $\bar{\Delta}$ in the simple asymptotic scale deformation introduced in [2]. Regrouping the terms and rescaling both parameter and phase variables will lead to a generic asymptotic expansion in a new asymptotic scale deformation $\mathcal{W}^*$, that is very similar to the one that has been used for a saddle loop [3]. It will permit to easily prove the existence of at least $2k-2$ zeroes $\bar{x}$ close to $\bar{x}=0$ for $\varepsilon=0$, that can be lifted to $2k-2$ different zeroes in x for $\varepsilon \neq 0$ (Proposition 2.1). This procedure will now be described.

After substitution of (3) in (2), we can regroup the terms in the asymptotic expansion (2) of $\bar{\Delta}$ as follows:

$$\bar{\Delta}(x,\nu,\varepsilon) = \beta(\nu,\varepsilon) - u(\nu,\varepsilon)x^{1+\varepsilon\alpha} + x^{1+\varepsilon\alpha}\left[\sum_{i=1}^{k-1} x^i\left(\eta_i^{(2)}(\nu,\varepsilon) - \big(1+\varepsilon u(\nu,\varepsilon)\big)\eta_i^{(1)}(\nu,\varepsilon)x^{i\varepsilon\alpha}\right) + \Psi_k(x,\nu,\varepsilon)\right], \tag{4}$$

where Ψ_k is a C^∞ function for $x>0$ sufficiently small; it is C^k for $x \geqslant 0$ and for any $0<\delta<1$:

$$\Psi_k(x,\nu,\varepsilon) = \mathrm{O}\big(x^{k-\delta}\big), \quad x \downarrow 0. \tag{5}$$

In a natural way, we introduce a sequence of new compensators ω_i, $i \geqslant 1$, defined as: $\omega_i(x,\varepsilon) = \omega(x, i\varepsilon\alpha)$, $\forall i \geqslant 1$, where $\omega = \omega_1$ is the traditional compensator associated to the saddles s_1 and s_2:

$$\omega(x,\varepsilon\alpha) = \frac{x^{\varepsilon\alpha}-1}{\varepsilon\alpha} \quad \text{if } \varepsilon\alpha \neq 0 \quad \text{and} \quad \omega(x,0) = \log x.$$

Then, expression (4) reduces to

$$\bar{\Delta}^1(x,\nu^1,\varepsilon) = \bar{\Delta}\big(x, P_1(\nu^1,\varepsilon)\big) = \beta + x^{1+\varepsilon\alpha}\left[-u + \sum_{i=1}^{k-1}\big(-\varepsilon\hat{\eta}_i^{(1)}x^i\omega_i + \hat{\eta}_i^{(2)}x^i\big) + \Psi_k\big(x, P_1(\nu^1,\varepsilon)\big)\right], \tag{6}$$

where P_1^{-1} is the transformation $(\nu,\varepsilon) \mapsto (\nu^1,\varepsilon)$ with $\nu^1 = (\beta, u, \hat{\eta}^{(1)}_{1\ldots k-2}, \hat{\eta}^{(2)}_{1\ldots k-2})$ defined by

$$\begin{cases} \hat{\eta}_i^{(1)} = i\alpha(1+\varepsilon u)\eta_i^{(1)}, \\ \hat{\eta}_i^{(2)} = \eta_i^{(2)} - (1+\varepsilon u)\eta_i^{(1)}. \end{cases} \tag{7}$$

Next, the phase variable x is rescaled by $x = \varepsilon^2\bar{x}$. For $\bar{\omega}_i(\bar{x},\varepsilon) = \omega_i(\varepsilon^2\bar{x},\varepsilon)$, there exists an analytic function $\varphi_i : \mathbb{R} \to \mathbb{R}$ with $\varphi_i(0) = 0$ such that

$$\overline{\omega_i}\big(\bar{x},\varepsilon\big) = \big(1 + 2i\alpha\varepsilon\log\varepsilon\big(1+\varphi_i(\varepsilon\log\varepsilon)\big)\big)\omega_i(\bar{x},\varepsilon) + 2\log\varepsilon\big(1+\varphi_i(\varepsilon\log\varepsilon)\big).$$

Then, expression (6) reduces to

$$\begin{aligned}\bar{\Delta}^2(\varepsilon^2\bar{x},\nu^2,\varepsilon) = \bar{\Delta}^1\big(\varepsilon^2\bar{x}, P_2(\nu^2,\varepsilon)\big) &= \beta - u\varepsilon^{2+2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha} \\ &\quad + \varepsilon^{2+2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha}\sum_{i=1}^{k-1}\big[-\varepsilon^{2i+1}\hat{\gamma}_i^{(1)}\bar{x}^i\omega_i(\bar{x},\varepsilon) + \varepsilon^{2i}\hat{\gamma}_i^{(2)}\bar{x}^i\big] \\ &\quad + \varepsilon^{2+2\varepsilon\alpha}\bar{x}^{1+\varepsilon\alpha}\Psi_k\big(\varepsilon^2\bar{x}, P_1\circ P_2(\nu^2,\varepsilon)\big)\end{aligned} \tag{8}$$

by means of a transformation $P_2^{-1}:(\nu^1,\varepsilon)\mapsto(\nu^2,\varepsilon)$, where $\nu^2=(\beta,u,\hat\gamma^{(1)}_{1\ldots k-2},\hat\gamma^{(2)}_{1\ldots k-2})$ and

$$\begin{cases}\hat\gamma_i^{(1)}(\hat\eta_i^{(1)},\varepsilon)=\hat\eta_i^{(1)}\bigl(1+2i\alpha\varepsilon\log\varepsilon\bigl(1+\varphi_i(\varepsilon\log\varepsilon)\bigr)\bigr),\\ \hat\gamma_i^{(2)}(\hat\eta_i^{(1)},\hat\eta_i^{(2)},\varepsilon)=\hat\eta_i^{(2)}-2\hat\eta_i^{(1)}\varepsilon\log\varepsilon\bigl(1+\varphi_i(\varepsilon\log\varepsilon)\bigr).\end{cases}\tag{9}$$

Finally, the parameter ν^2 is rescaled by a map $S:(\mathbb{R}^{2k-1},0)\to(\mathbb{R}^{2k-2},0)$:

$$(\bar\nu,\varepsilon)=\bigl(\bar\beta,\bar u,\bar\eta^{(1)}_{1\ldots k-2},\bar\eta^{(2)}_{1\ldots k-2},\varepsilon\bigr)\mapsto\nu^2=\bigl(\beta,u,\hat\gamma^{(1)}_{1\ldots k-2},\hat\gamma^{(2)}_{1\ldots k-2}\bigr),\tag{10}$$

with

$$\begin{cases}\beta=\varepsilon^{2k}\bar\beta,\\ u=\varepsilon^{2k-2}\bar u,\\ \hat\gamma_i^{(1)}=\varepsilon^{2k-2i-3}\bar\eta_i^{(1)},\ \forall 1\leqslant i\leqslant k-2,\\ \hat\gamma_i^{(2)}=\varepsilon^{2k-2i-2}\bar\eta_i^{(2)},\ \forall 1\leqslant i\leqslant k-2.\end{cases}\tag{11}$$

In this way, a map $\varXi$ is introduced:

$$\bar\Delta\bigl(\varepsilon^2\bar x,P_1\circ P_2(S(\bar\nu,\varepsilon),\varepsilon)\bigr)=\varepsilon^{2k}\varXi(\bar x,\bar\nu,\varepsilon),\tag{12}$$

having an asymptotic expansion of order $2k-1$,

$$\begin{aligned}\varXi(\bar x,\bar\nu,\varepsilon)={}&\bar\beta-\bar u\varepsilon^{2\varepsilon\alpha}\bar x^{1+\varepsilon\alpha}+\varepsilon^{2\varepsilon\alpha}\bar x^{1+\varepsilon\alpha}\sum_{i=1}^{k-2}\bigl(-\bar\eta_i^{(1)}\bar x^i\omega_i(\bar x,\varepsilon)+\bar\eta_i^{(2)}\bar x^i\bigr)\\ &-\varepsilon\hat\gamma^{(1)}_{k-1}\varepsilon^{2\varepsilon\alpha}\bar x^{k+\varepsilon\alpha}\omega_{k-1}(\bar x,\varepsilon)+\hat\gamma^{(2)}_{k-1}\varepsilon^{2\varepsilon\alpha}\bar x^{k+\varepsilon\alpha}+\varepsilon^{2\varepsilon\alpha}\bar x^{1+\varepsilon\alpha}\bar\Psi_k(\bar x,\bar\nu,\varepsilon),\end{aligned}\tag{13}$$

in the scale $\mathcal{W}^*=\{1,\varepsilon^{2\varepsilon\alpha}\bar x^{1+\varepsilon\alpha},\varepsilon^{2\varepsilon\alpha}\bar x^{2+\varepsilon\alpha}\omega_1,\varepsilon^{2\varepsilon\alpha}\bar x^{2+\varepsilon\alpha},\ldots,\varepsilon^{2\varepsilon\alpha}\bar x^{i+\varepsilon\alpha}\omega_{i-1},\varepsilon^{2\varepsilon\alpha}\bar x^{i+\varepsilon\alpha},\ldots;i\geqslant 1\}$, where $\omega_i=\omega_i(\bar x,\varepsilon)$, $\forall i\geqslant 1$. The C^k map $\bar\Psi_k$ is defined by $\Psi_k(\varepsilon^2\bar x,P_1\circ P_2(\nu^2,\varepsilon))=\varepsilon^{2k-2}\bar\Psi_k(\bar x,\bar\nu,\varepsilon)$ and it satisfies for any $0<\delta<1$:

$$\bar\Psi_k(\bar x,\bar\nu,\varepsilon)=\mathrm{O}\bigl(\bar x^{k-\delta}\bigr),\quad \bar x\to 0.\tag{14}$$

Notice that the second genericity condition implies that $\hat\gamma^{(j)}_{k-1}(0)\neq 0$, $j=1,2$. Since each compensator ω_i behaves under derivation by the Euler operator ($\nabla=\bar x\frac{\partial}{\partial\bar x}$) exactly as the traditional compensator ω, up to some constants, it can be proven that $\mathcal{W}^*$ is a simple asymptotic scale deformation of the logarithmic scale $\mathcal{L}^*=\{1,\bar x,\bar x^i\log\bar x,\bar x^i\colon\ i\geqslant 2\}$, in the sense defined in [2]. Recall that a sequence of smooth functions $\mathcal{F}=\{f_i:]0,h_0[\to\mathbb{R}\colon\ i\in\mathbb{N}\}$ is said to have the Chebychev property [3], if $\forall n\in\mathbb{N},\forall a=(a_0,a_1,\ldots,a_{n-1})\in\mathbb{R}^n$, the function $f_a=\sum_{i=0}^{n-1}a_if_i+f_n$ has at most n zeroes in $[0,h_n]$, for a certain $0<h_n<h_0$.

The restriction of $\varXi$ to $\varepsilon=0$, $\varXi|_{\varepsilon=0}$, has a generic expansion of codimension $l\equiv 2k-2$ in the restricted logarithmic scale (i.e. $\varXi|_{\varepsilon=0}$ can generically be expanded in $1,\bar x,\bar x^i\log\bar x,\bar x^i,\bar x^k$: $2\leqslant i\leqslant k-1$, up to a remainder term $\bar\Psi_k$ with property (14)). Since $\mathcal{L}^*$ has the Chebychev property, it then follows that the maximal number of zeroes $\varXi|_{\varepsilon=0}$ bifurcating from $(\bar x,\bar\nu)=(0,0)$, is equal to l [3]. In fact, there exist sequences $(\bar x_n^i)_{n\in\mathbb{N}}$, $1\leqslant i\leqslant l$, and $(\bar\nu_n)_{n\in\mathbb{N}}$ such that

$$\forall 1\leqslant i\leqslant l\colon\quad \lim_{n\to\infty}\bar x_n^i\to 0\quad\text{and}\quad\lim_{n\to\infty}\bar\nu_n\to 0$$

and $0<\bar x_n^1<\cdots<\bar x_n^l$ are simple zeroes of $\varXi(\cdot,\bar\nu_n,0)$, i.e. $\forall n\in\mathbb{N}$, $\forall 1\leqslant i\leqslant l$:

$$\varXi(\bar x_n^i,\bar\nu_n,0)=0\quad\text{and}\quad\frac{\partial}{\partial\bar x}\varXi(\bar x_n^i,\bar\nu_n,0)\neq 0.$$

However, it is not immediate to transfer this result that holds for $\varepsilon=0$ to the required result on cyclicity for the particular family $(X_\lambda)_\lambda$, since we obtained the result after blowing up. Besides the calculations in [2], we also need the following proposition, leading to the required result.

Proposition 2.1. *Let* $R, E > 0$, *let* W *be a neighbourhood of* $\bar{\nu} = 0$ *in* $\mathbb{R}^l$ *and let* $\Xi :]0, R[\times W \times [0, E[\to \mathbb{R}$ *be a continuous map such that* $\frac{\partial \Xi}{\partial \bar{x}} :]0, R[\times W \times [0, E[\to \mathbb{R}, \bar{x} \in]0, R[$ *is a well-defined continuous map. If there exist sequences* $(\bar{\nu}_n)_{n\in\mathbb{N}}$ *in* W *and* $(\bar{x}_n^i)_{n\in\mathbb{N}}$ *in* $]0, R[$, $1 \leqslant i \leqslant l$ *such that* $\lim_{n\to\infty} \bar{\nu}_n = 0$ *and* $\forall 1 \leqslant i \leqslant l : \lim_{n\to\infty} \bar{x}_n^i = 0$ *and such that* $\forall n \in \mathbb{N}$*:* $\bar{x}_n^1 < \cdots < \bar{x}_n^l$ *are simple zeroes of* $\Xi(\cdot, \bar{\nu}_n, 0)$*. Then, there exist sequences* $(\varepsilon_n)_{n\in\mathbb{N}}$ *in* $]0, E[$ *and* $(\xi_n^i(\varepsilon_n))_{n\in\mathbb{N}}$ *in* $]0, R[$, $1 \leqslant i \leqslant l$ *such that* $\forall n \in \mathbb{N}$:

$$0 < \xi_n^1(\varepsilon_n) < \cdots < \xi_n^l(\varepsilon_n) \quad \text{are simple zeroes of } \Xi(\cdot, \bar{\nu}_n, \varepsilon_n)$$

and such that $\lim_{n\to\infty} \varepsilon_n = 0$ *and* $\lim_{n\to\infty} \xi_n^i(\varepsilon_n) = 0$ $(1 \leqslant i \leqslant l)$.

Proof. By the implicit function theorem, there exist, $\forall n \in \mathbb{N}$, a constant $E_n > 0$ and continuous curves ξ_n^i, $1 \leqslant i \leqslant l$, $\xi_n^i : [0, E_n] \to]0, \infty[: \varepsilon \mapsto \xi_n^i(\varepsilon)$ such that $\forall 1 \leqslant i \leqslant l$: $\xi_n^i(0) = \bar{x}_n^i$, and such that $\forall \varepsilon \in [0, E_n]$: $0 < \xi_n^1(\varepsilon) < \cdots < \xi_n^l(\varepsilon)$ and $\forall 1 \leqslant i \leqslant l$: $\Xi(\xi_n^i(\varepsilon), \bar{\nu}_n, \varepsilon) = 0$ and $\frac{\partial \Xi}{\partial \bar{x}}(\xi_n^i(\varepsilon), \bar{\nu}_n, \varepsilon) \neq 0$. Then, for each $n \in \mathbb{N}$, we can take an $\varepsilon_n > 0$ with $\varepsilon_n \downarrow 0$ for $n \to \infty$, such that $\xi_n^i(\varepsilon_n) \to 0$, for $n \to \infty$. This ends the proof, since we have found the required sequences.

Part of the complete elaboration can be found in [1] and an extensive paper is in preparation.

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