Available online at www.sciencedirect.com
SCIENCE DIRECT®
ELSEVIER
C. R. Acad. Sci. Paris, Ser. I 340 (2005) 503–506
COMPTES RENDUS MATHEMATIQUE
http://france.elsevier.com/direct/CRASS1/

Geometry

Constructing spherical CR manifolds by gluing tetrahedra

Elisha Falbel

Institut de mathématiques, analyse algébrique, 175, rue du Chevaleret, 75005 Paris, France

Received 29 December 2004; accepted 5 February 2005

Available online 5 March 2005

Presented by Étienne Ghys

Abstract

We propose a general method of constructing spherical CR manifolds by gluing tetrahedra adapted to CR geometry. We obtain spherical CR structures on the complement of the figure eight knot and the Whitehead link complement with holonomy in $\mathrm{PU}(2,1,\mathbb{Z}[\omega])$ and $\mathrm{PU}(2,1,\mathbb{Z}[\mathrm{i}])$ respectively (the same integer rings appearing in real hyperbolic geometry). ***To cite this article: E. Falbel, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Structures CR sphériques par recollement de tétrahèdres. On propose une méthode de construction géométrique des variétés CR sphériques par recollement des tétrahèdres. Pour les complémentaires de la figure huit et l'entrelac de Whitehead, on obtient des structures avec holonomies dans $\mathrm{PU}(2,1,\mathbb{Z}[\omega])$ et $\mathrm{PU}(2,1,\mathbb{Z}[\mathrm{i}])$ respectivement (les mêmes anneaux d'entiers que dans le cas hyperbolique réel). ***Pour citer cet article : E. Falbel, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

1. Introduction

Among the first seminal examples of complete hyperbolic manifolds is the complement of the figure eight knot. It was shown by Riley in [3] that the fundamental group of that manifold had a discrete representation contained in $\mathrm{PSL}(2,\mathbb{Z}[\omega])$ where $\mathbb{Z}[\omega]$ is the ring of Eisenstein integers. On the other hand the construction by Thurston [6] is based on gluing ideal tetrahedra and that led to general constructions on a large family of 3-manifolds. It is not known which hyperbolic manifolds admit a spherical CR structure. In fact very few constructions of spherical CR 3-manifolds with discrete holonomy exist at all. The only construction of such a structure on a hyperbolic 3-manifold which is not a Seifert manifold previous to this work is essentially for the Whitehead link and other

E-mail address: falbel@math.jussieu.fr (E. Falbel).

1631-073X/$ – see front matter
doi:10.1016/j.crma.2005.02.014

manifolds obtained from it by Dehn surgery in [4,5]. We propose in this Note a general method of construction analogue to Thurston's in the real hyperbolic case.

1.1. Complex hyperbolic space and its boundary (see [2] as a general reference)

Let $\mathbb{C}^{2,1}$ denote the complex vector space equipped with the Hermitian form

$$\langle z, w\rangle = z_1\overline{w}_3 + z_2\overline{w}_2 + z_3\overline{w}_1.$$

Consider the following subspaces in $\mathbb{C}^{2,1}$: $V_0 = \{z \in \mathbb{C}^{2,1} \setminus \{0\}\colon \langle z, z\rangle = 0\}$, $V_- = \{z \in \mathbb{C}^{2,1}\colon \langle z, z\rangle < 0\}$. Let $P : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{C}P^2$ be the canonical projection onto complex projective space. Then $\mathbf{H}^2_{\mathbb{C}} = P(V_-)$ equipped with the Bergman metric is complex hyperbolic space. The boundary of complex hyperbolic space is $P(V_0) = \partial\mathbf{H}^2_{\mathbb{C}}$. The isometry group $\widehat{\mathbf{PU}(2, 1)}$ of $\mathbf{H}^2_{\mathbb{C}}$ comprises holomorphic transformations in $\mathbf{PU}(2, 1)$, the unitary group of $\langle\cdot, \cdot\rangle$, and anti-holomorphic transformations arising from elements of $\mathbf{PU}(2, 1)$ followed by complex conjugation. A manifold modeled on the boundary of complex hyperbolic space is called a *Spherical CR manifold*. The *Heisenberg group* $\mathfrak{N}$ is identified with the set of pairs $(z, t) \in \mathbb{C} \times \mathbb{R}$. Using stereographic projection, we can identify $\partial\mathbf{H}^2_{\mathbb{C}}$ with the one-point compactification $\overline{\mathfrak{N}}$ of $\mathfrak{N}$. A point $p = (z, t)$ in the Heisenberg group and the point ∞ are lifted to the following points in $\mathbb{C}^{2,1}$:

$$\hat{p} = \begin{bmatrix} \frac{-|z|^2 + \mathrm{i}t}{2} \\ z \\ 1 \end{bmatrix} \quad \text{and} \quad \hat{\infty} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Given any three points p_1, p_2, p_3 in $\partial\mathbf{H}^2_{\mathbb{C}}$ we define *Cartan's angular invariant* $\mathbb{A}$ as

$$\mathbb{A}(p_1, p_2, p_3) = \arg\bigl(-\langle\hat{p}_1, \hat{p}_2\rangle\langle\hat{p}_2, \hat{p}_3\rangle\langle\hat{p}_3, \hat{p}_1\rangle\bigr).$$

We define $\mathbb{C}$-*circles* in $\partial\mathbf{H}^2_{\mathbb{C}}$ to be the boundaries of complex geodesics in $\mathbf{H}^2_{\mathbb{C}}$. Given two points p_1 and p_2 in Heisenberg space, we write $[p_1, p_2]$ for a choice of one of the two segments of $\mathbb{C}$-circle joining them. The choice will be determined from the context.

1.2. Tetrahedra

Definition 1.1. A symmetric tetrahedron is a configuration of four points with an anti-holomorphic symmetry and a choice of $\mathbb{C}$-circle segments joining each pair of points.

By normalizing the coordinates of the four points we can assume that they are given by

$$p_1 = \infty, \quad p_2 = 0, \quad q_1 = (1, t_3), \quad q_2 = (z, t_4)$$

with $t_4 = t_3|z|^2$ (cf. [2,7]). The symmetry interchanges $p_1 \to p_2$ and $q_1 \to q_2$ simultaneously. In that case $\mathbb{A}(p_1, p_2, q_1) = \mathbb{A}(p_1, p_2, q_2)$ and $\mathbb{A}(p_1, q_1, q_2) = \mathbb{A}(p_2, q_1, q_2)$. To each vertex of a tetrahedron we associate the complex coordinates of the three vertical lines obtained when we place that vertex at ∞. That gives us four Euclidean triangles. If $p_1 = \infty$, $p_2 = 0$, $q_1 = (1, t_3)$, $q_2 = (z, t_4)$, the invariant of the triangle determined by the points (p_2, q_1, q_2) at the line determined by p_2 is z.

We consider Fig. 1 to describe the parameters of a tetrahedron. Note that contrary to the ideal tetrahedron in real hyperbolic geometry the Euclidean invariant at each vertex is not the same. The following proposition follows immediately from the considerations above.

Proposition 1.2. *For a symmetric tetrahedron given by*

$$p_1 = \infty, \quad p_2 = 0, \quad q_1 = (1, t), \quad q_2 = \bigl(z, |z|^2 t\bigr)$$

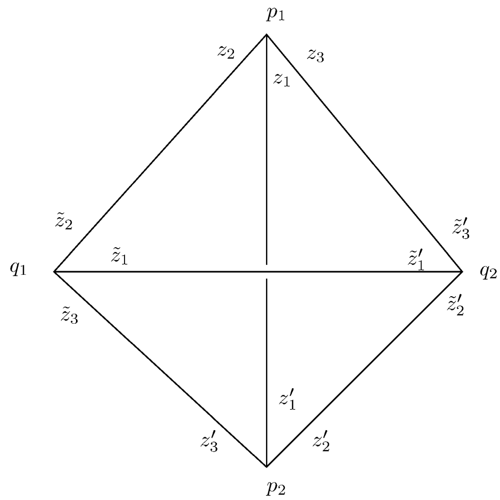

Fig. 1. Parameters for a CR tetrahedron.

then $z_1 = z$, $z'_1 = \frac{z}{|z|^2}$, $\tilde z_1 = z\frac{(\bar z-1)(1-\mathrm{i}t)}{(z-1)(1+\mathrm{i}t)}$ *and* $\tilde z'_1 = \frac{\tilde z_1}{|\tilde z_1|^2}$*, where, as usual,* $z_2 = \frac{1}{1-z_1}$ *and* $z_3 = 1 - \frac{1}{z_1}$*.*

$$\operatorname{tg}\mathbb{A}(p_1,p_2,q_1) = -\mathrm{i}\frac{z_1\bar z_1 - z_1 - z_1\tilde z_1 + \tilde z_1}{z_1\bar z_1 - z_1 + z_1\tilde z_1 - \tilde z_1}, \qquad \operatorname{tg}\mathbb{A}(p_1,q_1,q_2) = -\mathrm{i}\frac{z_1\bar z_1 + z_1 - z_1\tilde z_1 - \tilde z_1}{z_1\bar z_1 - z_1 + z_1\tilde z_1 - \tilde z_1}.$$

2. Gluing tetrahedra

We will only give more details for the figure eight knot construction. We make $\omega = \mathrm{e}^{-\mathrm{i}\pi/3}$. If $p_1 = (0, 2+\sqrt{3})$, $p_2 = (0, -(2+\sqrt{3}))$, $q_1 = (\omega, 0)$ and $q_2 = (1, 0)$ then in the parameters above $z_1 = \tilde z_1 = \overline{\omega}$. Moreover the tetrahedron is symmetric and $A(q_1, q_2, p_2) = \frac{\pi}{3}$ and $A(p_1, q_2, p_2) = -\frac{\pi}{3}$. This can be thought of as a regular tetrahedron, although there is not a permutation group acting on the tetrahedron.

We define the procedure of filling the faces from the one skeleton of the tetrahedra in such a way that the 2-skeleton will be $\mathbb{Z}_2$-invariant: Taking $\mathbb{C}$-segments from p_1 to the edges $[q_1, q_2]$, $[q_2, p_2]$ and $\mathbb{C}$-segments from p_2 to the edges $[q_1, q_2]$, $[q_1, p_1]$. Observe that the rays start from p_1 or p_2 and not from q_1 or q_2.

Each of those triangles is part of a $\mathbb{C}$-sphere (see [1]). The object defined by the above procedure is homeomorphic to a tetrahedron.

Theorem 2.1. *There exists a spherical CR-structure on the complement of the figure eight knot with discrete holonomy contained in* $\mathrm{PU}(2,1,\mathbb{Z}[\omega])$*. Moreover, the holonomy of the torus link is faithful parabolic.*

We use the same identifications that Thurston used in his construction for a hyperbolic real structure on the figure eight knot. We realize the two tetrahedra in the Heisenberg space gluing a pair of sides. The side pairings transformations are shown in Fig. 2 where the two tetrahedra are represented with a common side (here we introduce the point $q_3 = (\bar\omega, 0)$). They are determined by their action on three points and are defined by:

$$g_1 : (q_2, q_1, p_1) \to (q_3, p_2, p_1),$$
$$g_2 : (p_2, q_1, q_2) \to (p_1, q_3, q_2),$$
$$g_3 : (q_1, p_2, p_1) \to (q_2, p_2, q_3).$$

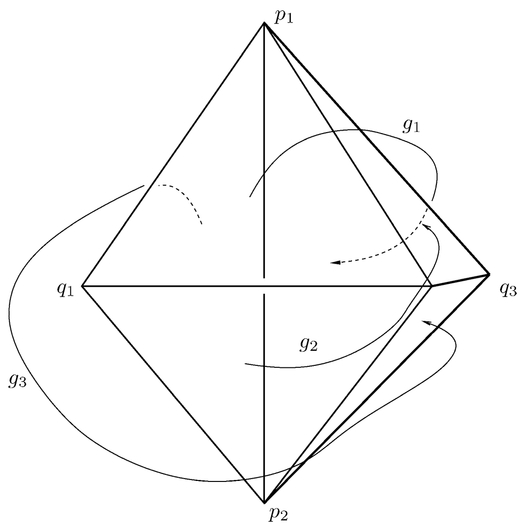

Fig. 2. Identifications on the tetrahedra.

A computation shows that the generators are in the Eisenstein–Picard group after a conjugation by an element of $\mathrm{PU}(2,1)$ bringing p_1 to ∞.

Acknowledgements

We thank R. Benedetti, M. Deraux, J.-P. Koseleff, J. Paupert and P. Will for many fruitful discussions.

References

[1] E. Falbel, V. Zocca, A Poincaré's polyhedron theorem for complex hyperbolic geometry, J. Reine Angew. Math. 516 (1999) 133–158.
[2] W.M. Goldman, Complex Hyperbolic Geometry, Oxford Math. Monographs, Oxford University Press, 1999.
[3] R. Riley, A quadratic parabolic group, Math. Proc. Cambridge Philos. Soc. 77 (1975) 281–288.
[4] R. Schwartz, Real hyperbolic on the outside, complex hyperbolic on the inside, Invent. Math. 151 (2) (2003) 221–295.
[5] R. Schwartz, Spherical CR Geometry and Dehn Surgery Research Monograph, 2004.
[6] W. Thurston, The geometry and topology of 3-manifolds, Lecture notes, 1979.
[7] P. Will, Lagrangian decomposability of some two-generator subgroups of $\mathrm{PU}(2,1)$, C. R. Acad. Sci. Paris, Ser. I 340 (5) (2005) 353–358.