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Probability Theory

Convergence in law for certain additive functionals of symmetric stable processes under strong topology

Mohamed Ait Ouahra

Departement of Mathematics, Faculty of Sciences Semlalia, Cadi Ayyad University, B.P. 2390, Marrakech 40.001, Morocco

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Abstract

We give some limit theorems of certain additive functionals for symmetric stable process of index $1 < \alpha \leqslant 2$ in anisotropic Besov space. These results generalize those obtained by Eisenbaum (1997) and by Csaki et al. (2002). ***To cite this article: M. Ait Ouahra, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Convergence en loi pour certaines fonctionnelles additives d'un processus stable symétrique sous une topologie forte. Nous donnons certains théorèmes limites pour certaines fonctionnelles additives d'un processus stable symétrique d'indice $1 < \alpha \leqslant 2$ dans une classe d'espace de Besov anisotropique. Ces résultats généralisent ceux obtenus par Eisenbaum (1997) et par Csaki et al. (2002). ***Pour citer cet article : M. Ait Ouahra, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Version française abrégée

Soit $\{X_t,\ t \geqslant 0\}$ un processus de Lévy stable symétrique d'indice $1 < \alpha \leqslant 2$ i.e. un processus càdlàg à accroissements indépendants stationnaire et de fonction caractéristique : $\mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda X_t} = \mathrm{e}^{-t|\lambda|^\alpha}$ pour tout $\lambda \in \mathbb{R}$.

Il est bien connu d'après Boylan [4], que la mesure $\mu_t(.)$ définite par $\mu_t(A) = \int_0^t \mathbf{1}_A(X_s)\,\mathrm{d}s$ admet une densité notée L_t^x par rapport à la mesure de Lebesgue ; $(L_t^x,\ t \geqslant 0,\ x \in \mathbb{R})$ est appelé la famille des temps locaux associé à X, de plus L_t^x admet une version *p.s.* continue (en t et x).

En notant $(p_t(x),\ x \in \mathbb{R},\ t \geqslant 0)$ les densités de transition de X, on pose : $c_\alpha = \int_0^{+\infty} (p_t(0) - p_t(1))\,\mathrm{d}t$. Nous appelons le drap Brownien fractionnaire d'indice $H \in [0,1]$ un processus gaussien centré continu $(B_t(x);\ x \in \mathbb{R},\ t \geqslant 0)$ de covariance $\mathbb{E}B_s(x)B_t(y) = (s \wedge t)\frac{1}{2}(|x|^H + |y|^H - |x-y|^H)$. Nous savons (cf. Ei-

E-mail address: ouahra@ucam.ac.ma (M. Ait Ouahra).

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senbaum [7] que pour : $y_1, y_2, \ldots, y_n$ n réels distincts, on a :

$$\left(X, \frac{1}{\varepsilon^{(\alpha-1)/2}}\left(L_t^{\varepsilon x+y_k} - L_t^{y_k}\right),\ x \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ t \geqslant 0\right) \xrightarrow[\varepsilon\to 0]{\text{loi}} \left(X, \sqrt{c_\alpha} B^{[y_k]}_{2L_t^{y_k}},\ x \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ t \geqslant 0\right),$$

où $\{B_t^{[y_k]}(x);\ x \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ t \geqslant 0\}$ est un système gaussien indépendant de X, composé de n draps Browniens fractionnaires d'indice $(\alpha - 1)$, tous indépendants.

Dans le cas $\alpha = 2$ (i.e. X est un mouvement Brownien) on retrouve le résultat de Yor [13]. D'une part pour $\alpha = 2$ Csaki et al. [6] ont montré la convergence en loi suivante :

$$\left\{X_t, \frac{H_t^{(1)}(\varepsilon x) - H_t^{(1)}(0)}{\pi \varepsilon^{1/2}},\ x \in \mathbb{R},\ t \geqslant 0\right\} \xrightarrow{\text{loi}} \left\{X_t, \mathbb{B}^{\mathcal{H}}(x, L_t^0),\ x \in \mathbb{R},\ t \geqslant 0\right\},$$

où $\mathbb{B}^{\mathcal{H}}(\cdot,\cdot)$ est un drap Brownien independent de X et $H_t^{(1)}(\cdot)$ est la derivée fractionnaire d'ordre $\gamma = 0$ du temps local de X que nous allons définir dans la suite.

L'étude classique de la convergence en loi des suites de processus stochastiques utilise essentiellement deux cadres fonctionnels qu'on peut trouver par exemple dans Billingsley [5]. Lorsque les trajectoires ont des discontinuités, on a recours à l'espace de Skorohod $\mathcal{D}([0,1])$ ou $\mathcal{D}(\mathbb{R}^+)$ muni de la topologie de Skorohod. Lorsqu'elles sont continues, on se contente géneralement de l'espace $\mathcal{C}([0,1])$ ou $\mathcal{C}(\mathbb{R}^+)$, muni de la topologie de la convergence uniforme (sur les compacts).

Lamperti [10] est le premier à s'écarter de ce point de vue classique. En faisant observer que cette approche n'exploite pas pleinement la régularité des trajectoires du mouvement Brownien, il étend aux espaces de Hölder le principe d'invariance pour les lignes polygonales établi par Donsker et Prohorov dans $\mathcal{C}[0,1]$. L'intérêt d'adapter le cadre fonctionnel à la régularité du processus est très bien exposé dans l'introduction de Lamperti [10]. C'est dans ce contexte que se situe cette note ; plus précisement nous allons établir le résultat de Eisenbaum [7] et de Csaki et al. [6] dans une classe d'espaces de Besov anisotropiques.

1. Introduction

Let $\{X_t,\ t \geqslant 0\}$ be a symmetric stable process with an index $\alpha \in (1,2]$. This means that X is a real-valued process with stationary independent increment such that,

$$\mathbb{E}\,\mathrm{e}^{\mathrm{i}\lambda X_t} = \mathrm{e}^{-t|\lambda|^\alpha}, \quad \lambda \in \mathbb{R}.$$

This process admits a continuous local time process $(L_t^x,\ x \in \mathbb{R},\ t \geqslant 0)$ (see for instance Boylan [4]).

A fractional Brownian sheet with index $H \in [0,1]$ is a continuous centered Gaussian process $(B_t(x),\ x \in \mathbb{R},\ t \geqslant 0)$ with covariance

$$\mathbb{E}\big(B_s(x)B_t(y)\big) = (s \wedge t)\frac{1}{2}\big(|x|^H + |y|^H - |x-y|^H\big).$$

Let $(p_t(x),\ x \in \mathbb{R},\ t \geqslant 0)$ denote the transition density of X and put $c_\alpha = \int_0^{+\infty}(p_t(0) - p_t(1))\,\mathrm{d}t$.

The following weak convergence was obtained by Eisenbaum [7]: For $y_1, y_2, \ldots, y_n$, n distinct reals

$$\left(X, \frac{1}{\varepsilon^{(\alpha-1)/2}}\left(L_t^{\varepsilon x+y_k} - L_t^{y_k}\right),\ x \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ t \geqslant 0\right) \xrightarrow[\varepsilon\to 0]{\text{law}} \left(X, \sqrt{c_\alpha} B^{[y_k]}_{2L_t^{y_k}},\ x \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ t \geqslant 0\right),$$

where $\{B_t^{[y_k]}(x), x \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ t \geqslant 0\}$ is a Gaussian system independent of X, composed of n fractional Brownian sheets with index $(\alpha - 1)$.

Let us note that, in the case $\alpha = 2$, which is the case when X is a Brownian motion, Csaki et al. [6] showed that

$$\left\{X_t, \frac{H_t^{(1)}(\varepsilon x) - H_t^{(1)}(0)}{\pi \varepsilon^{1/2}}\right\} \xrightarrow{\text{law}} \left\{X_t,\ \mathbb{B}^{\mathcal{H}}(x, L_t^0)\right\}, \tag{1}$$

where $\mathbb{B}^{\mathcal{H}}(\cdot,\cdot)$ is a standard Brownian sheet independent of X and $H_t^{(1)}(\cdot)$ is a fractional derivative of order $\gamma = 0$ of Brownian local time (see definition below).

Note also that all the above results have been established in the space of continuous functions.

The classical framework of limit theorems is the space of continuous functions $\mathcal{C}([0,1])$ equipped with the uniform convergence topology, for continuous process, and the space $\mathcal{D}([0,1])$ endowed with the Skorohod's topology for discontinuous ones.

The purpose of this Note is to extend Eisenbaum's [7] limit theorem and Csaki et al. [6] result to anisotropic Besov spaces (see Theorem 4.1 and 4.2 below).

This Note is organized as follows: in Section 2, we begin by stating some necessary notations; in particular, we recall a tightness criterion in anisotropic Besov space. In Section 3, we prove the regularity of fractional derivatives of local times of stable process. In the last section, we prove our main results.

2. Anisotropic Besov space

For any function $f : \mathbf{I} =: [0,1]^2 \to \mathbb{R}$, any $h \in \mathbb{R}$, and $i = 1, 2$; the progressive difference in direction e_i (where $e_i = (\delta_{1,i}, \delta_{2,i})$ denotes the ith coordinate vector in $\mathbb{R}^2$), is defined by:

$$\Delta_{h,i} f(z) = \begin{cases} f(z + h\cdot e_i) - f(z) & \text{if } z, z + he_i \in \mathbf{I}, \\ 0 & \text{otherwise.} \end{cases}$$

For any $\bar{h} = (h_1, h_2) \in \mathbb{R}^2$, we set:

$$\Delta_{\bar{h}} f = \Delta_{h_1,1} \circ \Delta_{h_2,2} f.$$

Now, for any Borel function $f : \mathbf{I} \to \mathbb{R}$, such that $f \in L^p(\mathbf{I})$, $1 \leqslant p < \infty$ or $f \in \mathcal{C}(\mathbf{I})$ if $p = \infty$, one can measure its smoothness by its modulus of continuity computed in $L^p(\mathbf{I})$ norm. To this end, let us define for any $t \in [0,1]$ and $\bar{t} = (t_1, t_2) \in \mathbf{I}$:

$$\begin{cases} \omega_{p,i}(f,t) = \sup_{|h| \leqslant t} \|\Delta_{h,i} f\|_p, \quad i = 1, 2, \\ \omega_p(f, \bar{t}) = \sup_{|h_1| \leqslant t_1, |h_2| \leqslant t_2} \|\Delta_{\bar{h}} f\|_p. \end{cases}$$

For $\delta = (\delta_1, \delta_2)$, $0 < \delta_1, \delta_2 < 1$ and $\beta \in \mathbb{R}$, we will consider the real valued function $\omega_\beta^\delta(.)$ defined on $\mathbf{I}$ by:

$$\omega_\beta^\delta(t_1, t_2) = t_1^{\delta_1} t_2^{\delta_2} \left(1 + \log \frac{1}{t_1 t_2}\right)^\beta.$$

We are now going to consider some anisotropic generalized Hölder classes in L^p-norm.
Define the norm:

$$\|f\|_p^{\omega_\beta^\delta} := \|f\|_{L^p(\mathbf{I})} + \sup_{0<t\leqslant 1} \frac{\omega_{p,1}(f,t)}{\omega_\beta^\delta(1,t)} + \sup_{0<t\leqslant 1} \frac{\omega_{p,2}(f,t)}{\omega_\beta^\delta(t,1)} + \sup_{0<t_1,t_2\leqslant 1} \frac{\omega_p(f,\bar{t})}{\omega_\beta^\delta(t_1,t_2)}.$$

Definition 2.1. Let $1 \leqslant p \leqslant \infty$, $\delta = (\delta_1, \delta_2)$, $0 < \delta_1, \delta_2 < 1$ and $\beta \in R$; the *anisotropic Besov spaces* are defined as follows: $\mathrm{Lip}_p(\delta, \beta) := \{f \in L^p(\mathbf{I}) \colon \|f\|_p^{\omega_\beta^\delta} < \infty\}$.

$\mathrm{Lip}_p(\delta, \beta)$ endowed with the norm $\|\cdot\|_p^{\omega_\beta^\delta}$ is a non-separable Banach space. We will consider a separable Banach subspace of $Lip_p(\delta, \beta)$ defined by:

$$\mathrm{lip}_p^*(\delta,\beta) := \big\{ f \in \mathrm{Lip}_p(\delta,\beta) \colon \omega_{p,i}(f,t) = \mathrm{o}\big(\omega_\beta^\delta(t,1)\big),\ i = 1,2 \text{ as } t \to 0; \\ \text{and } \omega_p(f,t_1,t_2) = \mathrm{o}\big(\omega_\beta^\delta(t_1,t_2)\big) \text{ as } t_1 \wedge t_2 \to 0 \big\}.$$

For detailed discussion on anisotropic Besov spaces see Kamont [8,9].

In the sequel it is more convenient to work with $\mathrm{lip}_p^*(\delta,\beta)$ instead of $\mathrm{Lip}_p(\delta,\beta)$. As the canonical injection of $\mathrm{lip}_p^*(\delta,\beta)$ in $\mathrm{Lip}_p(\delta,\beta)$ is continuous, weak convergence in the former implies weak convergence in the latter. Let us recall the following theorem due to Prohorov, which plays a crucial role in the sequel.

Theorem 2.2. *Let $\mathcal{S}$ be a separable, complete metric space. A family Π of probability measures in $\mathcal{S}$ is relatively compact if and only if it is tight.*

A sufficient conditions for the tightness in $\mathrm{lip}_p^*(\delta,\beta)$ is given by the following theorem

Theorem 2.3 (Boufoussi and Lakhel [3]). *Let $\{X^n_{s,t},\ (s,t)\in \mathbf{I}\}_{n\geqslant 1}$ be a sequence of stochastic processes satisfying*:

(a) $X^n_{\cdot,0}=X^n_{0,\cdot}=x\in R$.
(b) $\forall p\geqslant 2$, *there exists a positive constant C_p such that*

$$E|X^n_{s,t}-X^n_{s',t}-X^n_{s,t'}+X^n_{s',t'}|^p\leqslant C_p|s-s'|^{\delta_1 p}|t-t'|^{\delta_2 p},$$

for some $\delta=(\delta_1,\delta_2)$. Then the sequence $\{X^n\}_{n\geqslant 1}$ is tight in $\mathrm{lip}_p^*(\delta,\beta)$ for all $\beta>\frac{2}{p}$ and $p\geqslant 2$.

3. Regularity of local time and its fractional derivative

Throughout this Note we use $X=(\Omega,F,F_t,\mathbb{P})$ to denote the canonical realization of a real-valued symmetric stable process of index $\alpha\in\,]1,2]$. We will always denote by $(L_t^x,\ (t,x)\in\mathbb{R}^+\times\mathbb{R})$ its local time. The following lemma due to Ait Ouahra and Eddahbi [2] presents the mixed Hölder regularity of local time in x and in t.

Lemma 3.1. *Let $T>0$, there exists a constant $C>0$ such that for any $(s,t)\in[0,T]^2$, $|x|,|y|\leqslant M$ and integers $p\geqslant 2$, we have*

$$\|L_t^x-L_t^y-L_s^x+L_s^y\|_p\leqslant C|t-s|^{\frac{\alpha-1}{2\alpha}}|x-y|^{\frac{\alpha-1}{2}},$$

where $\|\cdot\|_p=[E|\cdot|^p]^{1/p}$ and C is a constant depending only of α and p.

This allows us to define the fractional derivative of order γ for $L_t^{\cdot}$ for all $0\leqslant\gamma<\frac{\alpha-1}{2}$. In this Note we use this notion in the case $\gamma=0$. For the other cases see for instance Samko et al. [12].

We define the fractional derivative of order 0 as follows: $D_\pm^0 f(x)=-\int_0^{+\infty}\frac{f(x\pm y)-\mathbf{1}_{\{0<y<1\}}f(x)}{y}\,\mathrm{d}y$ for $f\in\mathcal{C}^\beta\cap L^1(\mathbb{R})$, $\beta>0$ where $\mathcal{C}^\beta$ is the Hölder space of order β. Note that $D^0=D_+^0-D_-^0$ is the Hilbert transform (modulo a factor of $\frac{1}{\pi}$).

We define also $D^0L_t^{\cdot}(x):=H_t^{(1)}(x)=\int_0^t\frac{\mathrm{d}s}{X_s-x}$.

In order to prove Theorem 4.2, we need a regularity property. The following lemma is similar to Marcus and Rosen's result [11] (Lemma 3.3) but for fractional derivatives of local times instead of local times.

Lemma 3.2. *Let $T>0$ and $D\in\{D_+^0,D_-^0,D^0\}$. Then there exists a finite random variable $C>0$ such that for every $s,t\in[0,T]$, $x,z\in[-M,M]$ and integers $p\geqslant 2$, we have*

$$\left\|DL_t^{\cdot}(x)-DL_t^{\cdot}(z)\right\|_p\leqslant Ct^{\frac{\alpha-1}{2\alpha}}|x-z|^{\frac{\alpha-1}{2}}.$$

Proof. Let us give the proof for D_+^0; the other case can be derived similarly and by linearity.

From the definition of D_+^0 we have for all integers $p\geqslant 2$

$$J = \left\| D_+^0 L_t^{\cdot}(x) - D_+^0 L_t^{\cdot}(z) \right\|_p \leqslant \int_0^1 \frac{\|L_t^x - L_t^{x+y} - L_t^z + L_t^{z+y}\|_p}{y} \, \mathrm{d}y + \int_1^{+\infty} \frac{\|L_t^{z+y} - L_t^{x+y}\|_p}{y} \, \mathrm{d}y$$
$$=: J_1 + J_2.$$

Using the Lemma 3.1, we get $J_2 \leqslant Ct^{\frac{\alpha-1}{2\alpha}} |x-z|^{\frac{\alpha-1}{2}} \int_1^{+\infty} \frac{1}{y} \mathbf{1}_{A(x,z)}(y) \, \mathrm{d}y$ where $A(x,z) = \{y\colon\ |x+y| \leqslant C$ or $|z+y| \leqslant C\}$ has measure $\leqslant 4C$. Then $J_2 \leqslant Ct^{\frac{\alpha-1}{2\alpha}} |x-z|^{\frac{\alpha-1}{2}}$. Now, we deal with J_1.

$$J_1 \leqslant \int_0^b \frac{\|L_t^x - L_t^{x+y}\|_p + \|L_t^z - L_t^{z+y}\|_p}{y} \, \mathrm{d}y + \int_b^1 \frac{\|L_t^x - L_t^z\|_p + \|L_t^{x+y} - L_t^{z+y}\|_p}{y} \, \mathrm{d}y.$$

We consider the two cases $|x-z| > \frac{1}{e}$ and $|x-z| \leqslant \frac{1}{e}$.

- $|x-z| > \frac{1}{e}$
  Using Lemma 3.1 and choosing $\frac{1}{e} < b < |x-z|$, we have $J_1 \leqslant Ct^{\frac{\alpha-1}{2\alpha}} |x-z|^{\frac{\alpha-1}{2}}$.
- $|x-z| \leqslant \frac{1}{e}$

By choosing $0 < b < |x-z|$, Lemma 3.1 yields $J_1 \leqslant Ct^{\frac{\alpha-1}{2\alpha}} |x-z|^{\frac{\alpha-1}{2}}$.

Therefore, we deduce that $J \leqslant Ct^{\frac{\alpha-1}{2\alpha}} |x-z|^{\frac{\alpha-1}{2}}$.

Remark 1. The similar result for fractional derivative of order $0 < \gamma < \frac{\alpha-1}{2}$ is given in Ait Ouahra [1]; more precisely we have the following regularity $|D^\gamma L_t^{\cdot}(x) - D^\gamma L_t^{\cdot}(z)| \leqslant Ct^{(\frac{\alpha-1}{\alpha}-\gamma)\frac{1}{2}-\varepsilon} |x-z|^{(\frac{\alpha-1}{2}-\gamma)-\varepsilon}$.

4. Limit theorem

In this section we present two limit theorems. The first theorem is an application of Lemma 3.1 and the second is a consequence of Lemma 3.2.

Theorem 4.1. *Let $\delta = (\delta_1, \delta_2) > 0$, $p \geqslant 2$, $\beta > \frac{2}{p}$, and let X be a symmetric stable process with index $1 < \alpha \leqslant 2$. For $y_1, y_2, \ldots, y_n$ n distinct reals, we have*:

$$\left(X, \frac{L_t^{\varepsilon x + y_k} - L_t^{y_k}}{\varepsilon^{(\alpha-1)/2}};\ x \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ t \geqslant 0 \right) \xrightarrow{\text{law}} \left(X, \sqrt{c_\alpha} B^{[y_k]}_{2L_t^{y_k}}(x);\ x \in \mathbb{R},\ 1 \leqslant k \leqslant n,\ t \geqslant 0 \right).$$

The convergence holds in the separable Banach space $\mathrm{lip}_p^*((\frac{\alpha-1}{2\alpha}, \frac{\alpha-1}{2}), \beta)$.

Theorem 4.2. *Let B be a standard Brownian motion and l_t^x its local time. Then as ε goes to* 0,

$$\left\{ B_t, \frac{H_t^{(1)}(\varepsilon x) - H_t^{(1)}(0)}{\pi \varepsilon^{1/2}} \right\} \xrightarrow{\text{law}} \left\{ B_t, \mathbb{B}^{\mathcal{H}}(x, l_t^0) \right\},$$

with $\mathbb{B}^{\mathcal{H}}(x, l_t^0) = \int_{\mathbb{R}} \frac{1}{\pi} \log \frac{|y|}{|y-x|} \, \mathrm{d}_y \mathbb{B}(y, l_t^0)$ *a standard Brownian sheet, independent of* $\{B_t,\ t \geqslant 0\}$ *where* $\mathbb{B}(\cdot,\cdot)$ *denotes a Brownian sheet indexed by* $\mathbb{R} \times \mathbb{R}_+$. *This convergence holds in the anisotropic Besov space* $\mathrm{lip}_p^*((\frac{1}{4}, \frac{1}{2}), \beta)$ *for any* $p > \frac{2}{\beta}$.

Proof of Theorem 4.1. The convergence of the finite-dimensional distribution follows from the result of Eisenbaum [7]. It suffices to prove that the sequence of the process $A_t^\varepsilon(x) = \frac{1}{\varepsilon^{(\alpha-1)/2}}(L_t^{\varepsilon x+y_k} - L_t^{y_k})$ is tight in the anisotropic Besov space $\mathrm{lip}_p^*((\frac{\alpha-1}{2\alpha}, \frac{\alpha-1}{2}), \beta)$.

By virtue of the Lemma 3.1, we get $E|A_t^\varepsilon(x) - A_t^\varepsilon(z) - A_s^\varepsilon(x) + A_s^\varepsilon(z)|^{2m} \leqslant C|t-s|^{\frac{\alpha-1}{2\alpha}2m}|x-y|^{(\alpha-1)m}$. The tightness in the anisotropic Besov space $lip_p^*((\frac{\alpha-1}{2\alpha}, \frac{\alpha-1}{2}), \beta)$ is a consequence of the last inequality and Theorem 2.3. This complete the proof of Theorem 4.1. □

Proof of Theorem 4.2. Following the results of Csaki et al. [6], we have the convergence of the finite-dimensional distributions. To finish, we prove the tightness of $A_t^\varepsilon(x) := \frac{H_t^{(1)}(\varepsilon x) - H_t^{(1)}(0)}{\pi\varepsilon^{1/2}}$ in $\mathrm{lip}_p^*((\frac{1}{4}, \frac{1}{2}), \beta)$. By applying the Markov property of B at time s and Lemma 3.2, we get

$$\begin{aligned}
&\mathbb{E}\left|A_t^\varepsilon(x) - A_t^\varepsilon(y) - A_s^\varepsilon(x) + A_s^\varepsilon(y)\right|^{2m} = \mathbb{E}\left|H_{t-s}^{(1)}(\varepsilon x) - H_{t-s}^{(1)}(\varepsilon y)\right|^{2m} o\theta_s \\
&\quad = \mathbb{E}\left[\mathbb{E}\left[|H_{t-s}^{(1)}(\varepsilon x) - H_{t-s}^{(1)}(\varepsilon y)|^{2m} o\theta_s / \mathcal{F}_s\right]\right] = \int_{\mathbb{R}} \mathbb{P}[B_s \in \mathrm{d}z]\mathbb{E}\left|H_{t-s}^{(1)}(\varepsilon x - z) - H_{t-s}^{(1)}(\varepsilon y - z)\right|^{2m} \\
&\quad \leqslant C|t-s|^{\frac{1}{4}2m}|x-y|^{\frac{1}{2}2m},
\end{aligned}$$

which gives the tightness in the anisotropic Besov space $\mathrm{lip}_p^*((\frac{1}{4}, \frac{1}{2}), \beta)$ for all $\beta > 0$ and $p > \frac{2}{\beta}$.

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