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Numerical Analysis

On the non existence of monotone linear schema for some linear parabolic equations

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Abstract

In this Note, we present a result concerning the non existence of linear monotone schema with fixed stencil on regular meshes for some linear parabolic equation in two dimensions. The parabolic equations of interest arise from non isotropic diffusion modelling. A corollary is that no linear monotone 9 points-schemes can be designed for the one-dimensional heat equation emerged in the plane with an arbitrary direction of diffusion. Some applications of this result are provided: for the Fokker–Planck–Lorentz model for electrons in the context of plasma physics; all linear monotone scheme for the one-dimensional hyperbolic heat equation treated as a two-dimensional problem are not consistent in the diffusion limit for an arbitrary direction of propagation. We also examine the case of the Landau equation. *To cite this article: C. Buet, S. Cordier, C. R. Acad. Sci. Paris, Ser. I 340 (2005).*

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Résumé

Sur la non existence de schémas linéaires monotones pour certaines équations paraboliques linéaires. Dans cette Note, nous présentons un résultat de non existence de schémas linéaires monotones avec un stencil fixé sur un maillage carré pour certaines équations paraboliques en dimension 2. Les équations paraboliques que l'on considère proviennent de modèles de diffusion anisotrope. Une conséquence du résultat est qu'il n'existe pas de schémas linéaires monotones à neuf points pour l'équation de la chaleur monodimensionnelle immergée dans le plan, avec une direction arbitraire. Nous présentons quelques applications : à l'équation de Fokker–Planck–Lorentz pour les électrons dans le contexte de la physique des plasmas ; Un schéma linéaire monotone pour l'équation de la chaleur hyperbolique monodimensionnelle et traité comme un problème bidimensionnel ne peut pas être consistant dans la limite de diffusion pour une direction arbitraire de propagation. On examine aussi le cas de l'équation de Landau. *Pour citer cet article : C. Buet, S. Cordier, C. R. Acad. Sci. Paris, Ser. I 340 (2005).* © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.

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On s'intéresse à l'approximation numérique de l'équation parabolique de la forme (1).

Dans le cas où c = ab, cette équation représente l'équation de la chaleur monodimensionnelle immergée en dimension 2. La diffusion agit donc uniquement dans la direction (a, b). Il est bien connu que cette équation vérifie un principe du maximum : soit deux données initiales telles que $f_0 \ge g_0$ alors $f \ge g$.

On considère une grille de calcul cartésienne à mailles carrées de pas d'espace h.

Nous montrons alors (Proposition 2.2) que, pour un stencil fixe de la forme (5), il existe toujours des directions (a, b) de diffusion pour lesquelles on ne peut construire de schéma linéaire monotone. La preuve est basée sur l'analyse de l'erreur de consistance.

Ce résultat est encore valide pour une matrice de diffusion dépendant de la variable d'espace (Proposition 3.1) donc en particulier pour l'équation de Fokker–Planck–Lorentz (11) issue de la physique des plasmas et représentant un modèle simplifié des collisions avec les ions pour les électrons. Nous montrons de plus que, pour l'équation du télégraphe (13), il ne peut y avoir de schémas linéaires monotones consistant avec la limite de diffusion (1) avec c = ab, pour une direction arbitraire de propagation.

Nous indiquons enfin comment ce résultat pourrait signifier que pour l'équation de Fokker–Planck–Landau (14), il ne peut y avoir de schémas positif pour une discrétisation de la forme naturelle c'est à dire de schémas quadratiques positifs, ce qui justifierait l'emploi d'algorithmes vraiment non linéaires par exemple basés sur la forme dite logarithmique (16), voir par exemple [3].

1. Introduction

In this Note, we are interested in two-dimensional linear parabolic equations of the following form

$$\partial_t f - \frac{1}{2} \nabla \cdot K \nabla f = 0, \tag{1}$$

in \mathbb{R}^2 for the space variable i.e. with f(x, y). The matrix *K* is supposed symmetric positive which ensure that the solutions of the Cauchy problem for Eq. (1) verify the principle of maximum

$$f_0 \ge g_0 \Rightarrow f \ge g. \tag{2}$$

For a constant matrix

$$K = \begin{pmatrix} a^2 & c \\ c & b^2 \end{pmatrix},\tag{3}$$

with c = ab Eq. (1) is just a monodimensional heat equation with diffusion along the (a, b) direction.

2. Main result

Let us assume that the matrix K is constant positive i.e. the coefficients verify $c^2 \leq a^2b^2$. We consider Cartesian mesh of \mathbb{R}^2 of size h. The quadrature points are of the form $(x, y)_i = ih$ with $i = (i_1, i_2) \in \mathbb{Z}^2$ and we note f_i the approximated value of f at the points *ih*. We consider a generic linear scheme for Eq. (1) of the form

$$\frac{\mathrm{d}}{\mathrm{d}t}f_i = \frac{1}{h^2}\sum_{j\in S}a_j(h)f_{i+j} \tag{4}$$

with the function $a_i(h)$ continuous and S represents the stencil,

$$S = \{j \text{ such that } -N \leq j_k \leq N, \ k = 1, 2\},\tag{5}$$

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N is an integer which relies on the size of the stencil. N = 1 represents 9-points scheme and N = 2 a 25-points scheme.

All type of linear schemes for Eq. (1) such as finite volume, finite elements or finite differences schemes can be rewritten in the form (4).

We recall the definition of the consistency (see by example [8] or [6]):

Definition 2.1. Let us define the consistency error for the scheme (4) for a sufficiently smooth function f as

$$E(f,h) = \frac{1}{h^2} \left(\sum_{j \in S} a_j(h) \right) f + \frac{1}{h} \left[\left(\sum_{j \in S} j_1 a_j(h) \right) \partial_x f + \left(\sum_{j \in S} j_2 a_j(h) \right) \partial_y f \right] \\ + \left[\left(\sum_{j \in S} j_1^2 a_j(h) - a^2 \right) \partial_{xx} f + 2 \left(\sum_{j \in S} j_1 j_2 a_j(h) - c \right) \partial_{xy} f + \left(\sum_{j \in S} j_2^2 a_j(h) - b^2 \right) \partial_{yy} f \right] \\ + O(h).$$
(6)

A scheme is called consistent for Eq. (1) provided that the consistency error (6) is a o(1) i.e. tends to 0 as $h \rightarrow 0$.

Our main result concerning the monotonicity of the scheme (4) is the following:

Proposition 2.2. For every size of the stencil N, there is no linear monotone consistent scheme for a matrix K such that $N \min(a^2, b^2) < |c| \le |ab|$.

Proof. We consider only the case $0 \le a \le b$ and c > 0, the other cases are obtained by symmetry.

Using the definition of the consistency for some particular choice of f, the leading order term in (6) for constant function gives $\sum_{i \in S} a_i(h) = o(h^2)$. Then, using functions of the form Cx and Cy respectively, one gets

$$\sum_{j \in S} j_1 a_j(h) = o(h), \qquad \sum_{j \in S} j_2 a_j(h) = o(h).$$
(7)

When $h \rightarrow 0$, the zeroth order term in the consistency error gives necessarily

$$\sum_{j \in S} j_1^2 a_j(0) = a^2,$$
(8)

$$\sum_{j \in S} j_1 j_2 a_j(0) = c,$$
(9)

$$\sum_{j \in S} j_2^2 a_j(0) = b^2.$$
⁽¹⁰⁾

Subtracting $\frac{1}{N} \times (9)$ from (8) gives

$$\sum_{j \in S} j_1 \left(j_1 - \frac{j_2}{N} \right) a_j(0) = a^2 - \frac{c}{N}$$

Note that all terms $j_1(j_1 - \frac{j_2}{N})$ are positive and are not identically vanishing. On the other side, we have $a^2 - \frac{c}{N} < 0$. Thus, this implies that there is at least one index $j_0 \neq (0, 0)$ such that $a_{j_0}(0) < 0$.

By continuity $a_{j_0}(h) < 0$ for sufficiently small $h < h_0$. Thus, for all $h < h_0$, considering an initial data of the form $f_i = 0$ for any index *i* except at the point i_0 at which it is strictly positive, the solution of (4) is negative for *t* sufficiently small at point $i_0 - j_0$.

In conclusion, for $h < h_0$ the scheme is not positive and, since it is linear, it is not monotone. This ends the proof. \Box

Remark 1. In the case c = ab, there is no linear monotone consistent 9-points (N = 1) scheme except for a direction of diffusion aligned with the axis direction or with the principal diagonal i.e. $(a, b) \in \{(\pm 1, 0), (0, \pm 1), (\pm 1, \pm 1)\}$.

Always in the case c = ab, there is no 25-points linear monotone and consistent scheme if $4\min(a^2, b^2) < \max(a^2, b^2)$ and this is half of the possible directions. For the others cases, we have no response. Note that 25-points are rarely used for diffusion equations in 2-D.

Remark 2. Another way of to view this result is: if one wants to construct a monotone consistent scheme for an arbitrary matrix K one must consider a nonlinear scheme or use an infinite stencil, for example a stencil growing when refining the mesh size.

Remark 3. In the case c = ab, for the directions that are not concerned with Proposition 2.2, we can always construct a linear monotone and consistent scheme using points in the stencil which belongs to the direction $(\pm 1, \pm N)$ and $(\pm 1, \pm 1)$.

3. Applications

3.1. The Fokker–Planck–Lorentz equation for electrons

We consider now the case of K depending of the space variables x, y. A particular example of such an equation comes from the plasma physics: the Lorentz operator. Lorentz operators appear for example when considering elastic collisions of heavy particles (e.g. ions) against light ones (e.g. electrons). It is the first order term of the inter-species collision operator representing the collisions of the heavy particles on the light one. An asymptotic expansion in terms of the small mass ratio can be found in [4,5]. In this case, in 2-D, the matrix K is of the form

$$K(x, y) = \Psi\left(\frac{1}{x^2 + y^2}\right) \left(\mathbf{Id} - (x, y)^t \otimes (x, y) \right)$$
(11)

where Ψ is a positive function. This can be interpreted as the heat equation over each sphere centered at 0 and is also called Laplace–Beltrami operator. The natural way to discretize such an equation would to consider a spherical mesh. The question we shall address here is what happens when a Cartesian grid is used.

The result of Proposition 2.2 is still valid for the matrix, depending on the space variables:

Proposition 3.1. For a fixed stencil N, there is no linear monotone consistent scheme provided that there exists some open set in which the matrix K(x, y) is such that $N \min(a^2, b^2) < |c| \le |ab|$.

The result also holds for any mesh obtained using a smooth map from the uniform grid.

Proof. We consider only the case $0 \le a \le b$; the other cases are obtained by symmetry. Without loss of generality, we consider a point (x_0, y_0) which belongs to all the meshes and such that $\min(a^2, b^2) \le N|c|$ at this point.

We write the diffusion operator in the following (expanded) form.

$$\nabla \cdot K \nabla f = a^2 \partial_{xx} f + 2c \partial_{xy} f + b^2 \partial_{yy} f + d \partial_x f + e \partial_y f$$

the truncature error is as in the constant case, except that now the functions a_j depends now on x and y and relations (7) are replaced by

$$\sum_{j \in S} j_1 a_j(h) - d = o(h), \qquad \sum_{j \in S} j_2 a_j(h) - e = o(h)$$
(12)

but we still have relations (8)–10). At point (x_0, y_0) , we conclude as for the constant case.

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The matrix K of Fokker–Planck–Lorentz model, see (11), satisfies the hypothesis of Proposition 3.1. Thus, for the Lorentz equation on a square mesh, there is no linear consistent monotone scheme.

3.2. Asymptotic preserving schemes for the hyperbolic heat equation

Let us consider the following hyperbolic heat equation in 2-D:

$$\varepsilon \partial_t u + a \partial_x u + b \partial_y u = \frac{1}{\varepsilon} (v - u),$$

$$\varepsilon \partial_t v - a \partial_x v - b \partial_y v = \frac{1}{\varepsilon} (u - v)$$
(13)

with a and b constant such that $a^2 + b^2 = 1$. This is just the monodimensional hyperbolic heat equation along the direction (a, b). The monodimensional system is also referred to as telegraph equations or Goldstein–Taylor equation.

It is well known that the asymptotic regime, when $\varepsilon \to 0$ of this system is of the type (1) for the function f = u + v with the matrix K, c = ab. In 1-D, or with a direction aligned with one of the axes of the meshes in 2-D, it is possible to construct a monotone linear scheme which is 'asymptotic preserving', that is, in the limit $\varepsilon \ll 1$, the scheme gives a linear monotone discretization of the 1-D heat equation; see, for example, [7].

Let us now assume that one uses a first order monotone finite volume method on a square mesh of size h to solve (13). Naturally, this scheme is a 9-point schemes. In the diffusive regime, that is for $h \gg \varepsilon$ or for $\frac{h}{\varepsilon}$ fixed and $\varepsilon \ll 1$, one obtains, at most, a 25-point scheme for f = u + v, and this scheme is also monotone.

However, Proposition 2.2 implies that there is no monotone linear consistent scheme for Eq. (1) with a direction of diffusion (a, b) such that $4\min(a^2, b^2) < \max(a^2, b^2)$. Consequently, the scheme we used for the hyperbolic heat equation (13) cannot be asymptotic preserving for an arbitrary direction (a, b): it leads to a non consistent scheme for f = u + v if $4\min(a^2, b^2) < \max(a^2, b^2)$, or in other words, there is an isotropic numerical diffusion of size O(1) in these cases.

3.3. The Landau equation

Let us consider now the Fokker-Planck-Landau equation modeling self collisions for charged particles:

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \nabla_{v} \cdot \int_{v'} \phi(v - v')(f'\nabla_{v}f - f\nabla_{v'}f')\,\mathrm{d}v' \tag{14}$$

with $f' = f(v'), f = f(v), v, v' \in \mathbb{R}^3$ and

$$\phi(v - v') = \frac{1}{|v - v'|^3} (|v - v'|^2 \mathbf{Id} - (v - v') \otimes (v - v'))$$

is indeed symmetric and positive.

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Solutions of (14) are expected to be positive since f is the function of distribution of the charged particles. This equation can also be written as

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \nabla_v \cdot K(f)\nabla_v f + \nabla_v \cdot \left(C(f)f\right) \tag{15}$$

and the matrix K and the vector C (which are derivatives of the potentials of Rosenbluth), are defined by

$$K(f)(v) = \int_{v'} \phi(v - v') f' dv',$$

$$C(f)(v) = \int_{v'} \phi(v - v') \nabla_{v'} f' dv'$$

and the matrix K is indeed symmetric positive. The collision term is quadratic, thus a natural discretization of (14) is quadratic that is, at each point of the discretization, the discretized Landau operator reads:

$$\sum_{j\in S} \left(\sum_{k\in\mathbb{Z}^3} a_{i+j,k}(h) f_k\right) f_{i+j}.$$

Proposition 3.1 suggests that such a scheme could never be a positive scheme, see [1] for such an example. This would justify the use of the fully nonlinear writing of the Landau equation, the so called Log form

$$\frac{\mathrm{d}}{\mathrm{d}t}f = \nabla_{v} \cdot \int_{v'} ff' \phi(v - v') \left(\nabla_{v} \log(f) - f \nabla_{v'} \log(f')\right) \mathrm{d}v'$$
(16)

in order to derive a positive scheme (see [3,2]).

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