## Partial Differential Equations

# A fourth order uniformization theorem on some four manifolds with large total $Q$-curvature 

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#### Abstract

Given a four-dimensional manifold $(M, g)$, we study the existence of a conformal metric for which the $Q$-curvature, associated to a conformally invariant fourth-order operator (the Paneitz operator), is constant. Using a topological argument, we obtain a new result in cases which were still open. To cite this article: Z. Djadli, A. Malchiodi, C. R. Acad. Sci. Paris, Ser. I 340 (2005). © 2005 Académie des sciences. Published by Elsevier SAS. All rights reserved.


## Résumé

Un théorème d'uniformisation d'ordre 4 sur certaines variétés de dimension 4 à large $Q$-courbure totale. Etant donnée une variété riemannienne compacte de dimension 4 , on étudie l'existence d'une métrique conforme, pour laquelle la $Q$-courbure, associée à un opérateur d'ordre 4 (l'opérateur de Paneitz) est constante. En utilisant un argument topologique, nous obtenons des résultats nouveaux dans des cas auparavant encore ouverts. Pour citer cet article: Z. Djadli, A. Malchiodi, C. R. Acad. Sci. Paris, Ser. I 340 (2005).
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## Version française abrégée

Sur une variété $(M, g)$ de dimension 4, l'opérateur de Paneitz $P_{g}$ et la $Q$-courbure associée sont des extensions naturelles de l'opérateur de Laplace-Beltrami et la courbure de Gauss sur les surfaces, plus particulièrement en ce qui concerne les propriétés conformes. Une question basique est de trouver dans la classe conforme de $g$ une

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métrique à $Q$-courbure constante, ce qui conduit à la résolution d'une équation elliptique d'ordre 4 à non-linéarité exponentielle.

Ce problème a été entièrement résolu dans le cas où $\int_{M} Q_{g} \mathrm{~d} V_{g}<8 \pi^{2}$ par Chang et Yang [4] en utilisant une méthode de minimisation. Dans cette note nous présentons nos résultats dans certains cas où $\int_{M} Q_{g} \mathrm{~d} V_{g}>8 \pi^{2}$, en utilisant une méthode de mini-max.

## 1. Introduction

On four-dimensional manifolds, there exists a geometric quantity, the $Q$-curvature, which enjoys properties analogous to the Gauss curvature in dimension two. In particular, it is also related to a conformally invariant operator and, once integrated, it gives information on the geometry and the topology of the manifold. If Ric ${ }_{g}$ denotes the Ricci tensor of a four-manifold $(M, g)$ and $R_{g}$ the scalar curvature, the $Q$-curvature $Q_{g}$ of $(M, g)$ is defined by, see [2]

$$
\begin{equation*}
Q_{g}=-\frac{1}{12}\left(\Delta_{g} R_{g}-R_{g}^{2}+3\left|\operatorname{Ric}_{g}\right|^{2}\right) \tag{1}
\end{equation*}
$$

We point out that the above definition is not universally adopted and might sometimes differ by a factor 2 . The associated conformally invariant operator $P_{g}$, introduced first by Paneitz, see [10], is defined as

$$
\begin{equation*}
P_{g}(\varphi)=\Delta_{g}^{2} \varphi+\operatorname{div}\left(\frac{2}{3} R_{g} g-2 \operatorname{Ric}_{g}\right) \mathrm{d} \varphi \tag{2}
\end{equation*}
$$

where $\varphi$ is any smooth function on $M$. Given a conformal metric $\tilde{g}=\mathrm{e}^{2 w} g$, one has

$$
\begin{equation*}
P_{\tilde{g}}=\mathrm{e}^{-4 w} P_{g} ; \quad P_{g} w+2 Q_{g}=2 Q_{\tilde{g}} \mathrm{e}^{4 w} \tag{3}
\end{equation*}
$$

Also, if $W_{g}$ denotes the Weyl's tensor of $M$, there holds

$$
\begin{equation*}
\int_{M}\left(Q_{g}+\frac{\left|W_{g}\right|^{2}}{8}\right) \mathrm{d} V_{g}=4 \pi^{2} \chi(M) \tag{4}
\end{equation*}
$$

In particular, since $\left|W_{g}\right|^{2} \mathrm{~d} V_{g}$ is a pointwise conformal invariant, it follows that the integral of $Q_{g}$ over $M$ is also a conformal invariant, and we denote it by

$$
k_{P}=\int_{M} Q_{g} \mathrm{~d} V_{g}
$$

As for the Gauss curvature in two dimensions, it is natural to study the uniformization problem, that is to ask whether on a given 4-manifold $(M, g)$ there exists a conformal metric of constant $Q$-curvature. In view of (3), the problem is equivalent to solving the following non-linear partial differential equation

$$
\begin{equation*}
P_{g} u+2 Q_{g}=2 k_{P} \mathrm{e}^{4 u} \quad \text { on } M \tag{5}
\end{equation*}
$$

Solutions of (5) can be found as critical points of the Euler functional

$$
\begin{equation*}
I I(w)=\left\langle P_{g} w, w\right\rangle+4 \int_{M} Q_{g} w \mathrm{~d} V_{g}-k_{P} \log \int_{M} \mathrm{e}^{4 w} \mathrm{~d} V_{g} ; \quad w \in H^{2}(M) \tag{6}
\end{equation*}
$$

A first affirmative answer to the question was given by Chang and Yang [4] under the condition $k_{P}<8 \pi^{2}$ and assuming that $P_{g}$ is a non-negative operator whose kernel only consists of the constant functions. Under these conditions, from the following inequality due to Adams, [1], see also [4]

$$
\begin{equation*}
\log \int_{M} \mathrm{e}^{4(u-\bar{u})} \mathrm{d} V_{g} \leqslant C+\frac{1}{8 \pi^{2}}\left\langle P_{g} u, u\right\rangle \tag{7}
\end{equation*}
$$

the functional $I I$ is coercive on $H^{2}(M)$ (modulo constants) and solutions can be obtained via minimization. The same result was later extended to higher dimensions by Brendle, [3], via a flow approach. By a work of Gursky, [7], the above conditions are satisfied on four-manifolds of positive Yamabe class for which $k_{P}>0$.

In the present Note, we make a different assumption on $k_{P}$, namely we consider the case in which $k_{P} \in$ $\left(8 \pi^{2}, 16 \pi^{2}\right)$. Suppose $M=\Sigma_{1} \times \Sigma_{2}$, where $\Sigma_{1}, \Sigma_{2}$ are surfaces with genus $g_{1}, g_{2} \geqslant 2$, endowed with the Poincaré metric. Then, using the Gauss-Bonnet theorem, one easily finds that in this case $k_{P}=\frac{16 \pi^{2}}{3}\left(g_{1}-1\right)\left(g_{2}-1\right)$. Hence some perturbations of the metric on these manifolds, for small values of $g_{1}, g_{2}$, will satisfy the assumptions of our result, which is the following.

Theorem 1.1. Suppose $\operatorname{ker} P_{g}=\{$ constants $\}$, and assume that $k_{P} \in\left(8 \pi^{2}, 16 \pi^{2}\right)$. Then there exists a metric $\tilde{g}$ conformal to $g$ with constant (and positive) $Q$-curvature.

For reasons of brevity we give the proof only when $P_{g}$ has no negative eigenvalues, referring for complete one to [6], which contains the more general existence result:

Theorem 1.2. Suppose $\operatorname{ker} P_{g}=\{$ constants $\}$, and assume that $k_{P} \neq 8 k \pi^{2}$, for $k=1,2,3, \ldots$ Then there exists a metric $\tilde{g}$ conformal to $g$ with constant $Q$-curvature.

Our method is based on a minimax argument related to that in [5], and combined with the following analytic result, proved in [9].

Theorem 1.3. Suppose $\operatorname{ker} P_{g}=\{$ constants $\}$, and that $\left(u_{l}\right)_{l}$ is a sequence of solutions of

$$
\begin{equation*}
P_{g} u_{l}+2 Q_{l}=2 k_{l} \mathrm{e}^{4 u_{l}} \quad \text { on } M \tag{8}
\end{equation*}
$$

for which $\int_{M} \mathrm{e}^{4 u_{l}} \mathrm{~d} V_{g}=1$. Here $k_{l}=\int_{M} Q_{l} \mathrm{~d} V_{g}$ and we assume that $Q_{l} \rightarrow Q_{0}$ in $C^{0}(M)$ with $k_{0}:=\int_{M} Q_{0} \mathrm{~d} V_{g} \neq$ $8 k \pi^{2}$ for $k=1,2,3, \ldots$ Then $\left(u_{l}\right)_{l}$ is bounded in $C^{\alpha}(M)$ for any $\alpha \in(0,1)$.

## 2. Proof of Theorem 1.1 for $\boldsymbol{P}_{\boldsymbol{g}} \geqslant 0$

First we give a characterization of the functions $u \in H^{2}(M)$ on which the functional II attains large negative values. The next result is a particular case of Lemma 2.4 in [6].

Lemma 2.1. Under the assumptions of Theorem 1.1 (in the case $P_{g} \geqslant 0$ ), the following property holds. For any $\varepsilon>0$ and any $r>0$ there exists a large positive $L=L(\varepsilon, r)$ such that for every $u \in H^{2}(M)$ with $I I(u) \leqslant-L$ there exists $p_{u} \in M$ with

$$
\begin{equation*}
\int_{M \backslash B_{r}\left(p_{u}\right)} \mathrm{e}^{4 u} \mathrm{~d} V_{g}<\varepsilon \tag{9}
\end{equation*}
$$

Lemma 2.1 allows us to embed continuously suitable sublevels of $I I$ into $M$.
Lemma 2.2. There exists a large $L>0$ and a continuous map $\Phi$ from $\{I I \leqslant-L\} \subseteq H^{2}(M)$ into $M$.
Proof. Since the functional $I I$ is invariant under the translations in $w$, we can assume that the $H^{2}$ functions we are dealing with satisfy the volume normalization $\int_{M} \mathrm{e}^{4 w} \mathrm{~d} V_{g}=1$. By Whitney's theorem, there exists $m \in \mathbb{N}$ and a diffeomorphism $\Omega: M \rightarrow \mathcal{M}$, where $\mathcal{M}$ is a smooth submanifold of $\mathbb{R}^{m}$. First we define the map $\tilde{\Phi}: H^{2}(M) \rightarrow$
$\mathbb{R}^{m}$ by $\tilde{\Phi}(u)=\int_{M} \Omega(x) \mathrm{e}^{4 u(x)} \mathrm{d} V_{g}(x)$, which is continuous, as one can check using (7) and some elementary estimates. We are going to prove the following claim

$$
\begin{equation*}
\text { for every } \delta>0 \text { there exists } L_{\delta}>0 \text { such that } I I(u) \leqslant-L_{\delta} \operatorname{implies} \operatorname{dist}(\tilde{\Phi}(u), \mathcal{M})<\delta \tag{10}
\end{equation*}
$$

To prove this claim we let $\varepsilon=\frac{\delta}{2} \frac{1}{\operatorname{diam}(\mathcal{M})}, r=\frac{\delta}{2} \frac{1}{\|\mathrm{~d} \Omega\|}$, and we apply Lemma 2.1 with these values of $\varepsilon$ and $r$. Then, if $I I(u) \leqslant-L(\varepsilon, r)$, we obtain a point $p_{u}$ such that (9) holds. By our normalization we can write $\tilde{\Phi}(u)-\Omega\left(p_{u}\right)=\int_{B_{r}\left(p_{u}\right)}\left(\Omega(x)-\Omega\left(p_{u}\right)\right) \mathrm{e}^{4 u(x)} \mathrm{d} V_{g}(x)+\int_{M \backslash B_{r}\left(p_{u}\right)}\left(\Omega(x)-\Omega\left(p_{u}\right)\right) \mathrm{e}^{4 u(x)} \mathrm{d} V_{g}(x)$. This implies $\left\|\tilde{\Phi}(u)-\Omega\left(p_{u}\right)\right\| \leqslant r\|\mathrm{~d} \Omega\|+\varepsilon \operatorname{diam}(\mathcal{M}) \leqslant \delta$, and hence (10) follows. Choosing $\delta$ sufficiently small, there exists a continuous projection $T$ from a $\delta$-neighborhood of $\mathcal{M}$ onto $\mathcal{M}$. Now it is sufficient to define $L_{\delta}=L(\varepsilon, r)$ and $\Phi$ as

$$
\Phi(u)=T \circ \tilde{\Phi}(u) ; \quad u \in H^{2}(M), u \in\{I I \leqslant-L\}
$$

This map is clearly continuous, so the proof is concluded.
The next step consists in finding a map $(x, \lambda) \mapsto \varphi_{\lambda, x} \in H^{2}(M), \lambda>0$ and $x \in M$, on which image the functional II attains large negative values.

Proposition 2.3. For $\lambda \in \mathbb{R}$ sufficiently large, there exists a map $\varphi_{\lambda,}$. $M \rightarrow H^{2}(M)$ with the following properties
(a) $\operatorname{II}\left(\varphi_{\lambda, .}\right) \rightarrow-\infty$ as $\lambda \rightarrow+\infty$ uniformly on $M$;
(b) $\Phi \circ \varphi_{\lambda, .}$ is homotopic to the identity.

Proof. For $\delta>0$ small, consider a smooth cut-off function $\chi_{\delta}: \mathbb{R}_{+} \rightarrow \mathbb{R}$ satisfying the following properties

$$
\begin{cases}\chi_{\delta}(t)=t, & \text { for } t \in[0, \delta]  \tag{11}\\ \chi_{\delta}(t)=2 \delta, & \text { for } t \geqslant 2 \delta ; \\ \chi_{\delta}(t) \in[\delta, 2 \delta], & \text { for } t \in[\delta, 2 \delta]\end{cases}
$$

Then, given $x \in M$ and $\lambda>0$, we define the function $\varphi_{\lambda, x}: M \rightarrow \mathbb{R}$ by

$$
\begin{equation*}
\varphi_{\lambda, x}(y)=\log \left(\frac{2 \lambda}{1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(y, x))}\right) \tag{12}
\end{equation*}
$$

where $\operatorname{dist}(\cdot, \cdot)$ denotes the distance function on $M$. This implies immediately

$$
\int_{M} \exp \left(4 \varphi_{\lambda, x}(y)\right) \mathrm{d} V_{g}(y)=\int_{M}\left(\frac{2 \lambda}{1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(y, x))}\right)^{4} \mathrm{~d} V_{g}(y)
$$

We divide the above integral into the metric ball $B_{\delta}(x)$ and its complement. By construction of $\chi_{\delta}$, working in normal coordinates centered at $x$, we have (for $\delta$ sufficiently small)

$$
\begin{aligned}
& \int_{B_{\delta}(x)}\left(\frac{2 \lambda}{1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(y, x))}\right)^{4} \mathrm{~d} V_{g}(y)=\int_{B_{\delta}^{\mathbb{R}^{4}}(0)}(1+\mathrm{O}(\delta))\left(\frac{2 \lambda}{1+\lambda^{2}|y|^{2}}\right)^{4} \mathrm{~d} y \\
& \quad=\int_{B_{\lambda \delta}^{\mathbb{R}^{4}}(0)}(1+\mathrm{O}(\delta))\left(\frac{2}{1+|y|^{2}}\right)^{4} \mathrm{~d} y=(1+\mathrm{O}(\delta))\left(\frac{8}{3} \pi^{2}+\mathrm{O}\left(\frac{1}{\lambda^{4} \delta^{4}}\right)\right) .
\end{aligned}
$$

On the other hand, for $\operatorname{dist}(y, x) \geqslant \delta$ there holds $\left(\frac{2 \lambda}{1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(y, x))}\right)^{4} \leqslant\left(\frac{2 \lambda}{1+\lambda^{2} \delta^{2}}\right)^{4}$. Hence, from the last two formulas we deduce

$$
\begin{equation*}
\int_{M} \exp \left(4 \varphi_{\lambda, x}(y)\right) \mathrm{d} V_{g}(y)=\frac{8}{3} \pi^{2}+\mathrm{O}(\delta)+\mathrm{O}\left(\frac{1}{\lambda^{4} \delta^{4}}\right)+\mathrm{O}\left(\frac{2 \lambda}{1+\lambda^{2} \delta^{2}}\right)^{4} \tag{13}
\end{equation*}
$$

Next we estimate the term $\int_{M} Q_{g}(y) \varphi_{\lambda, x}(y) \mathrm{d} V_{g}(y)$. We note that $\varphi_{\lambda, x}=\log \frac{2 \lambda}{1+4 \delta^{2} \lambda^{2}}$ on $M \backslash B_{2 \delta}(x)$ and that $\log \frac{2 \lambda}{1+4 \delta^{2} \lambda^{2}} \leqslant \varphi_{\lambda, x} \leqslant \log 2 \lambda$ on $B_{2 \delta}(x)$. Writing $\int_{M} Q_{g}(y) \varphi_{\lambda, x}(y) \mathrm{d} V_{g}(y)=\log \frac{2 \lambda}{1+4 \delta^{2} \lambda^{2}} \int_{M} Q_{g} \mathrm{~d} V_{g}+$ $\int_{M} Q_{g}(y)\left(\varphi_{\lambda, x}(y)-\log \frac{2 \lambda}{1+4 \delta^{2} \lambda^{2}}\right) \mathrm{d} V_{g}(y)$, from the last three formulas it follows that

$$
\begin{equation*}
\int_{M} Q_{g}(y) \varphi_{\lambda, x}(y) \mathrm{d} V_{g}(y)=k_{P} \log \frac{2 \lambda}{1+4 \delta^{2} \lambda^{2}}+\mathrm{O}\left(\delta^{4} \log \left(1+4 \delta^{2} \lambda^{2}\right)\right) \tag{14}
\end{equation*}
$$

Finally we estimate $\int_{M} \varphi_{\lambda, x} P_{g} \varphi_{\lambda, x}$ for large values of $\lambda$. With elementary computations we find

$$
\begin{aligned}
& \nabla_{j} \varphi_{\lambda, x}(y)=-\frac{\lambda^{2} \nabla_{j} \chi_{\delta}^{2}(\operatorname{dist}(x, y))}{1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(x, y))} \\
& \nabla_{i} \nabla_{j} \varphi_{\lambda, x}(y)=-\frac{\lambda^{2} \nabla_{i} \nabla_{j} \chi_{\delta}^{2}(\operatorname{dist}(x, y))}{1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(x, y))}+\frac{\lambda^{4} \nabla_{i} \chi_{\delta}^{2}(\operatorname{dist}(x, y)) \nabla_{j} \chi_{\delta}^{2}(\operatorname{dist}(x, y))}{\left(1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(x, y))\right)^{2}}
\end{aligned}
$$

These equations imply in particular $\Delta \varphi_{\lambda, x}(y)=-\frac{\lambda^{2} \Delta \chi_{\delta}^{2}(\operatorname{dist}(x, y))}{1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(x, y))}+\frac{\lambda^{4}\left|\nabla \chi_{\delta}^{2}(\operatorname{dist}(x, y))\right|^{2}}{\left(1+\lambda^{2} \chi_{\delta}^{2}(\operatorname{dist}(x, y))\right)^{2}}$. For $\operatorname{dist}(x, y) \leqslant \delta$ there holds $\Delta \varphi_{\lambda, x}(y)=-4 \lambda^{2} \frac{2+\lambda^{2}|y-x|^{2}}{\left(1+\lambda^{2}|y-x|^{2}\right)^{2}}+\mathrm{O}\left(\frac{\delta^{4} \lambda^{4} \operatorname{dist}^{2}(x, y)}{\left(1+\lambda^{2} \operatorname{dist}^{2}(x, y)\right)^{4}}\right)$, and $\left|\nabla \varphi_{\lambda, x}\right|(y) \leqslant C \frac{\lambda^{2} \operatorname{dist}(x, y)}{1+\lambda^{2} \operatorname{dist}^{2}(x, y)}$. Using a change of variables as before, we get

$$
\begin{aligned}
& \int_{B_{\delta}(x)}\left(\Delta \varphi_{\lambda, x}(y)\right)^{2} \mathrm{~d} V_{g}(y)=16 \int_{B_{\lambda \delta}(0)} \frac{\left(2+|y|^{2}\right)^{2}}{\left(1+|y|^{2}\right)^{4}}+\mathrm{O}\left(\delta^{4}\right)=32 \pi^{2} \log (\lambda \delta)+\mathrm{O}\left(\delta^{4}\right) \\
& \int_{B_{\delta}(x)}\left|\nabla \varphi_{\lambda, x}\right|^{2} \leqslant \frac{C}{\lambda^{2}} \int_{B_{\lambda \delta}(0)} \frac{|y|^{2}}{\left(1+|y|^{2}\right)^{2}} \leqslant C \delta^{2}
\end{aligned}
$$

On the other hand for $\delta \leqslant \operatorname{dist}(x, y) \leqslant 2 \delta$ and $\lambda$ large there holds

$$
\left|\nabla \varphi_{\lambda, x}(y)\right| \leqslant C \frac{\lambda^{2} \delta}{1+\lambda^{2} \delta^{2}} \leqslant \frac{C}{\delta} ; \quad\left|\Delta \varphi_{\lambda, x}(y)\right| \leqslant \frac{C \lambda^{2}}{1+\lambda^{2} \delta^{2}}+\frac{C \lambda^{4} \delta^{2}}{\left(1+\lambda^{2} \delta^{2}\right)^{2}} \leqslant \frac{C}{\delta^{2}}
$$

Since $\varphi_{\lambda, x}$ is constant outside $B_{2 \delta}(x)$, from the last estimates and (2) we deduce $\int_{M} \varphi_{\lambda, x}(y) P_{g} \varphi_{\lambda, x}(y) \mathrm{d} V_{g}(y) \leqslant$ $32 \pi^{2} \log \lambda+C$. Finally, from (13), (14) and the last formula it follows that $I I\left(\varphi_{\lambda, x}\right) \leqslant 32 \pi^{2} \log \lambda-4 k_{P} \log \lambda+$ $C \delta^{4} \log \lambda+C \rightarrow-\infty$ as $\lambda \rightarrow+\infty$. This concludes the proof of (a).

The statement (b) is an easy consequence of the definition of $\Phi$ and (12).
We now define the minimax scheme which provides existence of solutions for (5). Let $\tilde{\sim} \tilde{M}$ denote the (contractible) cone over $M$, which can be represented as $\widetilde{M}=(M \times[0,1]) /(M \times\{1\})$, so $\partial \widetilde{M}=M \times\{0\} \simeq M$. Fixing $\lambda$ sufficiently large, we define the following set of maps

$$
\Theta_{\lambda}=\left\{\theta: \tilde{M} \rightarrow H^{2}(M): \theta \text { is continuous and } \theta(M \times\{0\})=\varphi_{\lambda, \cdot}\right\}
$$

Then we have the following properties.
Lemma 2.4. For every $\lambda$ large the set $\Theta_{\lambda}$ is non-empty and moreover, letting

$$
\bar{\Theta}_{\lambda}=\inf _{\theta_{\lambda} \in \Theta_{\lambda}} \sup _{m \in \widetilde{M}} I I\left(\theta_{\lambda}(m)\right), \quad \text { there holds } \bar{\Theta}_{\lambda}>-\infty
$$

Proof. To prove that $\Theta_{\lambda} \neq \emptyset$, we just notice that the following map

$$
\bar{\theta}(x, t)=\left\{\begin{array}{ll}
\varphi_{2(1-\lambda) t+\lambda, x}, & \text { for } t \in[0,1 / 2] ; \\
2\left(\left(2-\varphi_{1, x}\right) t+\left(\varphi_{1, x}-1\right)\right) & \text { for } t \in[1 / 2,1] ;
\end{array} \quad(x, t) \in \tilde{M}\right.
$$

belongs to $\Theta_{\lambda}$. Assuming by contradiction that $\bar{\Theta}_{\lambda}=-\infty$, we could apply Lemma 2.2 to obtain a continuous map $\Psi: \widetilde{M} \rightarrow M$, with $\left.\Psi\right|_{\partial \widetilde{M}}$ homotopic to the identity on $M$. But then the map $H: M \times[0,1] \rightarrow M$ defined as $H(\cdot, t)=\Psi(\cdot, t)$ would be an homotopy between a constant map and a map homotopic to the identity on $M$, which is impossible since $M$ is not contractible. It follows that $\bar{\Theta}_{\lambda}>-\infty$.

By classical arguments, the scheme described before yields a Palais-Smale sequence $\left(u_{l}\right)_{l}$. Because of the translation invariance of $I I$ we can assume that $\int_{M} \mathrm{e}^{4 u_{l}} \mathrm{~d} V_{g}=1$. We now use a procedure from [11], also used in [5,8] and [12]. For $\rho$ in a neighborhood of 1 , we define the functional $I I_{\rho}: H^{2}(M) \rightarrow \mathbb{R}$ in the following way $I I_{\rho}(u)=\left\langle P_{g} u, u\right\rangle+4 \rho \int_{M} Q_{g} \mathrm{~d} V_{g}-4 \rho k_{P} \log \int_{M} \mathrm{e}^{4 u} \mathrm{~d} V_{g}$, whose critical points give rise to solutions of the equation

$$
\begin{equation*}
P_{g} u+2 \rho Q_{g}=2 \rho k_{P} \mathrm{e}^{4 u} \quad \text { in } M \tag{15}
\end{equation*}
$$

One can then define the minimax scheme for different values of $\rho$ and prove boundedness of some Palais-Smale sequence for $\rho$ belonging to a set $\Lambda$ which is dense in some neighborhood of 1 , see [6]. This implies solvability of (15) for $\rho \in \Lambda$. We then apply Theorem 1.3 , with $Q_{l}=\rho_{l} Q_{g}$, where $\left(\rho_{l}\right)_{l} \subseteq \Lambda$ and $\rho_{l} \rightarrow 1$, obtaining a solution to (5).

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