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Partial Differential Equations

On logarithmic Sobolev inequalities for higher order fractional derivatives

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Abstract

On $\mathbb{R}^n$, we prove the existence of sharp logarithmic Sobolev inequalities with higher fractional derivatives. Let s be a positive real number. Any function $f \in H^s(\mathbb{R}^n)$ satisfies

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln\left(\frac{|f(x)|^2}{\|f\|_2^2}\right) dx + \left(n + \frac{n}{s}\ln\alpha + \ln\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|f\|_2^2 \leqslant \frac{\alpha^2}{\pi^s}\left\|(-\Delta)^{s/2} f\right\|_2^2$$

with $\alpha > 0$ be any number and where the operators $(-\Delta)^{s/2}$ in Fourier spaces are defined by $\widehat{(-\Delta)^{s/2}} f(k) := (2\pi|k|)^s \hat{f}(k)$. ***To cite this article: A. Cotsiolis, N.K. Tavoularis, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

Résumé

Sur les inégalités de Sobolev logarithmiques pour les dérivées fractionnelles d'ordre supérieur. Sur $\mathbb{R}^n$, on établi l'existence d'inégalités de Sobolev logarithmiques optimales pour les dérivées fractionnelles d'ordre supérieur. Soit s et α deux réels positifs. Pour toute fonction $f \in H^s(\mathbb{R}^n)$, on établit l'inégalité suivante :

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln\left(\frac{|f(x)|^2}{\|f\|_2^2}\right) dx + \left(n + \frac{n}{s}\ln\alpha + \ln\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right)\|f\|_2^2 \leqslant \frac{\alpha^2}{\pi^s}\left\|(-\Delta)^{s/2} f\right\|_2^2.$$

L'opérateur $(-\Delta)^{s/2}$ est defini dans les espaces de Fourier par $\widehat{(-\Delta)^{s/2}} f(k) := (2\pi|k|)^s \hat{f}(k)$. ***Pour citer cet article : A. Cotsiolis, N.K. Tavoularis, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***

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1. Introduction

Logarithmic Sobolev inequalities have a wide range of applications and have been extensively studied (see, for example, [9,4,5,8,1,11] and the references therein).

Let Δ be the Laplacian in $\mathbb{R}^n$ and let $\hat{f}(k)$ denote the Fourier transform of $f \in L^1(\mathbb{R}^n)$:

$$\hat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k x} f(x)\,dx.$$

The operators $(-\Delta)^{s/2}$ are defined in Fourier spaces (i.e. in spaces with functions which have Fourier transform such as $L^p(\mathbb{R}^n)$ for $1 \leqslant p \leqslant 2$) as multiplication by $(2\pi|k|)^s$, i.e.

$$\widehat{(-\Delta)^{s/2}} f(k) := \bigl(2\pi|k|\bigr)^s \hat{f}(k).$$

The space $H^s(\mathbb{R}^n)$ is endowed with the inner product

$$(f, g)_{H^s(\mathbb{R}^n)} = \int_{\mathbb{R}^n} \bar{\hat{f}}(k)\hat{g}(k)\bigl(1 + \bigl(2\pi|k|\bigr)^{2s}\bigr)\,dk$$

and

$$\|f\|^2_{H^s(\mathbb{R}^n)} := \int_{\mathbb{R}^n} \bigl|\hat{f}(k)\bigr|^2\bigl(1 + \bigl(2\pi|k|\bigr)^{2s}\bigr)\,dk.$$

The original form of the *logarithmic Sobolev inequality* (cf. [12]) is

$$\int_{\mathbb{R}^n} \bigl|g(x)\bigr|^2 \ln\biggl(\frac{|g(x)|^2}{\|g\|_2^2}\biggr)\,dm \leqslant \frac{1}{\pi}\int_{\mathbb{R}^n} \bigl|\nabla g(x)\bigr|^2\,dm,$$

where $dm = e^{-\pi|x|^2}\,dx$ is the Gauss measure and $\|g\|_2$ is, of course, the norm in $L^2(\mathbb{R}^n, dm)$.

Choosing $g(x) = e^{\pi|x|^2/2} f(x)$ in the above inequality, and considering an homothetic change of variables ($\alpha > 0$), we find ([10], Th. 8.14, p. 223):

$$\int_{\mathbb{R}^n} \bigl|f(x)\bigr|^2 \ln\biggl(\frac{|f(x)|^2}{\|f\|_2^2}\biggr)\,dx + n(1 + \ln\alpha)\|f\|_2^2 \leqslant \frac{\alpha^2}{\pi}\int_{\mathbb{R}^n} \bigl|\nabla f(x)\bigr|^2\,dx, \tag{1}$$

where the L^2 norm is with respect to Lebesgue measure.

The purpose of this Note is to give a generalization of inequality (1) with the operators $(-\Delta)^{s/2}$, $s > 0$.

Definition 1.1 [6]. We consider the operator semigroups $e^{-t(-\Delta)^s}$, $t > 0$, defined by its Fourier transform:

$$\bigl(e^{-t(-\Delta)^s} f\bigr)^\wedge(k) = e^{-t(2\pi|k|)^{2s}} \hat{f}(k).$$

2. Logarithmic Sobolev inequalities

Theorem 2.1. *Let f be any function in $H^s(\mathbb{R}^n)$ and let $\alpha > 0$ be any real number. Then*

$$\int_{\mathbb{R}^n} \bigl|f(x)\bigr|^2 \ln\biggl(\frac{|f(x)|^2}{\|f\|_2^2}\biggr)\,dx + \biggl(n + \frac{n}{s}\ln\alpha + \ln\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\biggr)\|f\|_2^2 \leqslant \frac{\alpha^2}{\pi^s}\bigl\|(-\Delta)^{s/2} f\bigr\|_2^2 \tag{2}$$

Proof. Let $F_s(x)$ be the function with Fourier transform $\widehat{F_s}(k) = \mathrm{e}^{-t(2\pi|k|)^{2s}}$. Then $\mathrm{e}^{-t(-\Delta)^s} f = F_s * f$. However, $\widehat{\widehat{F_s}}(x) = \widehat{\widehat{F_s}}(-x) = F_s(x)$ because $F_s \in L^2(\mathbb{R}^n)$.

By Young's inequality [3,10] we see that $\mathrm{e}^{-t(-\Delta)^s}$ maps $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ provided $p < q$. Indeed,

$$\left\| \mathrm{e}^{-t(-\Delta)^s} f \right\|_q = \|F_s * f\|_q = \|\widehat{\widehat{F_s}} * f\|_q \leqslant \left(\frac{C_r C_p}{C_q}\right)^n \|\widehat{\widehat{F_s}}\|_r \|f\|_p \tag{3}$$

with $1 + \frac{1}{q} = \frac{1}{r} + \frac{1}{p}$ and $C_p^2 = p^{1/p}/p'^{1/p'}$ (p' denotes the dual index of p).

$$\|\widehat{\widehat{F_s}}\|_r \leqslant C_{r'}^n \|\widehat{F_s}\|_{r'} \tag{4}$$

according to Hausdorff–Young's inequality with $r' = \frac{1}{1/p - 1/q} = \frac{pq}{q-p}$. Also

$$\begin{aligned} \|\widehat{F_s}\|_{r'}^{r'} &= \int_{\mathbb{R}^n} \mathrm{e}^{-t(2\pi|k|)^{2s} r'} \,\mathrm{d}k = \frac{2\pi^{n/2}}{\Gamma(n/2)} \int_0^\infty \mathrm{e}^{-t(2\pi\rho)^{2s} r'} \rho^{n-1} \,\mathrm{d}\rho \\ &= \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{\Gamma(\frac{n+2s}{2s})}{n} (2\pi)^{-n} \left(\frac{t}{1/p - 1/q}\right)^{-n/2s}. \end{aligned} \tag{5}$$

So,

$$\left\| \mathrm{e}^{-t(-\Delta)^s} f \right\|_q \leqslant \left(\frac{C_p}{C_q}\right)^n \left(\frac{t K_s}{1/p - 1/q}\right)^{-\frac{n}{2s}(\frac{1}{p} - \frac{1}{q})} \|f\|_p \tag{6}$$

where $K_s = (2\pi)^{2s} [\frac{2\pi^{n/2}}{\Gamma(n/2)} \frac{\Gamma(n+2s/2s)}{n}]^{-2s/n}$.

We set $q = 2$ and let

$$t = \alpha^2 \left(\frac{1}{p} - \frac{1}{2}\right) / \pi^s \to 0. \tag{7}$$

From (6) and (7) we obtain the inequality:

$$\|f\|_2^2 - \|f\|_p^2 + \left(1 - \left(\left(\frac{C_p}{C_2}\right)^n \left(\frac{K_s}{\pi^s} \alpha^2\right)^{-nt\pi^s/2s\alpha^2}\right)^2\right) \|f\|_p^2 \leqslant \|f\|_2^2 - \left\| \mathrm{e}^{-t(-\Delta)^s} f \right\|_2^2 \tag{8}$$

Note that the right-hand side of (8), when divided by $2t$, tends to $\|(-\Delta)^{s/2} f\|_2^2$ (Theorem announced in [6], the proof is in the Appendix below).

If $f \in C_c^\infty(\mathbb{R}^n)$ we have

$$\frac{\mathrm{d}}{\mathrm{d}p} \|f\|_p^2 |_{p=2} = \frac{1}{2} \int_{\mathbb{R}^n} |f(x)|^2 \ln\left(\frac{|f(x)|^2}{\|f\|^2}\right) \mathrm{d}x.$$

A straightforward computation leads to

$$\lim_{p \to 2} \frac{1 - ((C_p/C_2)^n (K_s/\pi^s \alpha^2)^{n(p-2)/4\pi})^2}{2 - p} = \frac{1}{2}\left(n + \frac{n}{s} \ln\alpha + \ln\frac{s\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2s})}\right). \tag{9}$$

Eqs. (8) and (9) prove inequality (2) for $f \in H^s(\mathbb{R}^n)$ by density [6,13]. Indeed $f^2(x)(\ln|f(x)|)_+$ is integrable according to the Sobolev theorem [2]. Let $\{f_i\} \subset C_c^\infty(\mathbb{R}^n)$ such that $f_i \to f$ in $H^s(\mathbb{R}^n)$ and a.e. $\int f_i^2(x)(\ln(f_i(x))_+ \to \int f^2(x)(\ln(f(x))_+$ according to the Lebesgue theorem. □

Remark 1. The constants in inequality (2) are the best one because we use sharp inequalities in the proof.

Remark 2. We have equality in (2) only for $s = 1$. Since for $s \neq 1$ H–Y inequality (4) is strict.

Remark 3. For $\ell \in \mathbb{N}$, we have seen (cf. [7]) that $\|\nabla^\ell f\|_2 = C\|(-\Delta)^{\ell/2} f\|_2$ where

$$C = 2^{-\ell} \prod_{h=-\ell}^{\ell-1} (n+2h)^{1/2} \left[\frac{\Gamma((n-2\ell)/2)}{\Gamma((n+2\ell)/2)}\right]^{1/2}.$$

Thus we can have the following logarithmic inequality:

$$\int_{\mathbb{R}^n} |f(x)|^2 \ln\left(\frac{|f(x)|^2}{\|f\|_2^2}\right) \mathrm{d}x + \left(n + \frac{n}{\ell}\ln\alpha + \ln\frac{\ell\Gamma(\frac{n}{2})}{\Gamma(\frac{n}{2\ell})}\right)\|f\|_2^2 \leqslant \frac{\alpha^2}{C^2\pi^\ell}\|\nabla^\ell f\|_2^2$$

with $\ell \in \mathbb{N}$ and C as given above.

Appendix. Proof of Theorem 1.2 announced in [6]

A function f is in $H^s(\mathbb{R}^n)$ if and only if it is in $L^2(\mathbb{R}^n)$ and $I_s^t(f) = \frac{1}{t}[(f,f) - (f, \mathrm{e}^{-t(-\Delta)^s} f)]$ is uniformly bounded and we have in which case $\sup_{t>0} I_s^t(f) = \lim_{t\to 0} I_s^t(f) = (f, (-\Delta)^s f)$.

Proof.

$$I_s^t(f) = \frac{1}{t}\left[(f,f) - \left(f, \mathrm{e}^{-t(-\Delta)^s} f\right)\right] = \frac{1}{t}\left[\int_{\mathbb{R}^n} |\hat{f}(k)|^2 \left(1 - \mathrm{e}^{-t(2\pi|k|)^{2s}}\right) \mathrm{d}k\right].$$

When we pass to the limit

$$\lim_{t\to 0} I_s^t(f) = \int_{\mathbb{R}^n} \left(2\pi|k|\right)^{2s} |\hat{f}(k)|^2 \,\mathrm{d}k = \left\|(-\Delta)^{s/2} f\right\|_2^2 \tag{10}$$

so if $f \in H^s(\mathbb{R}^n)$, the limit exists and (10) holds. If the limit exists for some $f \in L^2(\mathbb{R}^n)$, then $f \in H^s(\mathbb{R}^n)$. Moreover, $\sup_{t>0} I_s^t(f) = \lim_{t\to 0} I_s^t(f)$. □

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