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C. R. Acad. Sci. Paris, Ser. I 340 (2005) 93–98

COMPTES RENDUS MATHEMATIQUE

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Calculus of Variations

Non-differentiable functionals and singular sets of minima

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Received 2 November 2004; accepted 5 November 2004

Available online 21 December 2004

Presented by John M. Ball

Abstract

We provide bounds for the Hausdorff dimension of the singular set of minima of functionals of the type $\int_\Omega F(x, v, Dv)$, where F is only Hölder continuous with respect to the variables (x, v). ***To cite this article: J. Kristensen, G. Mingione, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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Résumé

Fonctionnelles non-différentiable et l'ensembles singulier des minima. Nous bornons la dimension de Hausdorff de l'ensemble singulier des minima de fonctionnelles du type $\int_\Omega F(x, v, Dv)$ oú F n'est Hölderienne que par rapport aux la variables (x, v). ***Pour citer cet article : J. Kristensen, G. Mingione, C. R. Acad. Sci. Paris, Ser. I 340 (2005).***
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Version française abrégée

Soit Ω un domaine régulier de $\mathbb{R}^n$, $n \geqslant 2$. Dans cette note nous nous intéressons à la régularité partielle des minima de fonctionnelles de la forme

$$\mathcal{F}[v] = \int_\Omega F\big(x, v(x), Dv(x)\big)\,\mathrm{d}x \tag{1}$$

qui sont définies sur $W^{1,p}(\Omega, \mathbb{R}^N)$, $N \geqslant 2$, $p > 1$. Dans le cas où la densité d'énergie $F = F(x, y, z)$ vérifie (2) (i.e., est Hölderienne d'exposant α par rapport aux variables (x, y) et est C^2 par rapport à z) il est connu qu'un minimum u de $\mathcal{F}$ est de classe $C^{1,\alpha}_{\mathrm{loc}}$ en dehors un fermé $\Sigma_u \subset \Omega$ de mesure de Lebesgue nulle [6,7]. L'estimation

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doi:10.1016/j.crma.2004.11.019

de la dimension de Hausdorff de Σ_u est un problème essentiel de la théorie de la régularité partielle des minima de $\mathcal{F}$ ([4], page 117). Plus précisement la question est de savoir si la dimension de Hausdorff de Σ_u est inferieure à n. Nous démontrons qu'il existe une constante $\delta > 0$ qui ne dépend que de F, α, n, N et p telle que $\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \min\{\alpha, \delta\}$ pour tout $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ minimum de $\mathcal{F}$.

La difficulté principale de notre analyse vient de la non différentiabilité de la densité F par rapport à y qui prévent l'existence de l'équation d'Euler–Lagrange associeé à $\mathcal{F}$. Nous développens une technique d'estimation de la dimension de Hausdorff de Σ_u qui n'utilise pas l'équation d'Euler–Lagrange associée à la fonctionnelle étudieé. Cette nouvelle technique est basée sur des estimations elliptiques dans les espace de Sobolev fractionnaires, un nouveau principe de comparaison et lemme de Gehring.

1. Singular set estimates

Consider an integral functional of the type (1), where $v \in W^{1,p}(\Omega, \mathbb{R}^N)$, $\Omega \subset \mathbb{R}^n$ is a bounded and open domain, $n, N \geqslant 2$, $p > 1$ and $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$. The regularity problem for minimizers of $\mathcal{F}$, i.e. maps $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ such that $\mathcal{F}[u] \leqslant \mathcal{F}[v]$ for all $v \in u + W_0^{1,p}(\Omega, \mathbb{R}^N)$, is classical in the Calculus of Variations. The reader is referred to the recent monograph of Giusti [7] for a comprehensive introduction to the subject.

We require the following list of properties of the energy density F:

$$\begin{cases} F(x, y, \cdot) \text{ is } C^2, \\ \nu|z|^p \leqslant F(x, y, z) \leqslant L\big(|z|^p + 1\big), \\ \nu\big(1 + |z|^2\big)^{(p-2)/2}|\lambda|^2 \leqslant \big\langle F_{zz}(x, y, z)\lambda, \lambda\big\rangle \leqslant L\big(1 + |z|^2\big)^{(p-2)/2}|\lambda|^2, \\ \big|F(x_1, y_1, z) - F(x_2, y_2, z)\big| \leqslant \tilde{L} \min\big\{1, |x_1 - x_2|^\alpha + |y_1 - y_2|^\alpha\big\}\big(|z|^p + 1\big), \end{cases} \tag{2}$$

where $z, \lambda \in \mathbb{R}^{nN}$, $x, x_1, x_2 \in \Omega$ and $y, y_1, y_2 \in \mathbb{R}^N$. Here we assume that the constants and exponents satisfy $p \geqslant 1$, $0 < \nu \leqslant L \leqslant \tilde{L}$ and $\alpha \in (0, 1]$. Note in particular that $(2)_4$ means that the function $(x, y) \mapsto F(x, y, z)/(|z|^p + 1)$, is Hölder continuous with exponent α, uniformly in z. The assumptions in (2) are standard in the regularity theory for minima of integral functionals $\mathcal{F}$ (see e.g. [4,7]).

In the *scalar case* $N = 1$, where the functional $\mathcal{F}$ is defined on real valued functions, assumptions (2) allow to prove full interior regularity: the gradient Du of a minimizer $u \in W^{1,p}(\Omega)$ is locally Hölder continuous in Ω (and in fact is as regular as F is). This is false in the *vectorial case* $n, N \geqslant 2$, where celebrated counterexamples [11,13] show that minimizers can be singular (e.g., Lipschitz, but not C^1). Instead, work has focused on proving so-called *partial regularity*: there exists an open subset $\Omega_u \subset \Omega$ such that $u \in C^{1,\sigma}_{\mathrm{loc}}(\Omega_u, \mathbb{R}^N)$ for some $\sigma > 0$ and $|\Omega \setminus \Omega_u| = 0$ (see [5–7]). The set

$$\Sigma_u := \Omega \setminus \Omega_u \tag{3}$$

is called the singular set of the minimizer u.

The main open problem in the theory concerns the size of the singular set Σ_u: *Is partial regularity in the sense that* $|\Sigma_u| = 0$ *optimal*? In case the answer is negative, the problem is to give bounds for the Hausdorff dimension of Σ_u; see question (a), page 117 of [4].

There are no previous results about the Hausdorff dimension of singular sets of minima for general integral functionals (1). The only such estimates in the literature concern minima of very special functionals, like certain quadratic functionals and functionals of the type

$$v \mapsto \int_\Omega a\big(x, v(x)\big)\big|Dv(x)\big|^p \, \mathrm{d}x$$

under suitable Hölder continuity and boundedness assumptions on the function $a : \Omega \times \mathbb{R}^N \to \mathbb{R}$, see [1,5].

When the Hölder continuity exponent α in (2) is strictly smaller than one, an obvious obstacle is the possible nondifferentiability of $F = F(x, y, z)$ with respect to y. This causes the functional $\mathcal{F}$ in (1) to be *non-differentiable*, and, crucially, the Euler–Lagrange system is not available. However, even when F is smooth the hypotheses (2) do not garantee that the Euler–Lagrange system $\operatorname{div} F_z(x, u, Du) = F_y(x, u, Du)$ is well-behaved (when $\alpha < 1$, no growth condition for F_y follows from (2) and it is not even clear if minimizers satisfy the Euler–Lagrange system). In fact, apart from some very special cases the usual methods for estimating the Hausdorff dimension of singular sets simply do not work.

In [8] we present a method that can overcome these difficulties and thereby answer the previous question showing that *partial regularity in the sense of* $|\Sigma_u| = 0$ *is never optimal*. Indeed, denoting by $\dim_{\mathcal{H}}(\Sigma_u)$ the Hausdorff dimension of Σ_u, we have the following:

Theorem 1.1. *Let* $u \in W^{1,p}(\Omega, \mathbb{R}^N)$ *be a minimizer of the functional* $\mathcal{F}$ *under the assumptions* (2)*. Then there exists a positive number* $\delta \equiv \delta(n, N, p, L/\nu) > 0$ *such that*

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \min\{\alpha, \delta\}. \tag{4}$$

The number δ is determined by use of Gehring's Lemma, which in a standard way yields higher integrability: if u is a minimizer of $\mathcal{F}$, then $Du \in L^q_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$ for some exponent $q \equiv q(n, N, p, L/\nu) > p$. We have $\delta := q - p$, and hence δ is non-increasing as a function of the growth-ellipticity ratio L/ν, and

$$\lim_{L/\nu \to \infty} \delta = 0. \tag{5}$$

It is therefore possible to give estimates for δ in terms of the data $(n, N, p, L/\nu)$: this requires a careful bookkeeping of the constants in the proof of Gehring's lemma, a strategy that has been followed when studying, for instance, quasiconformal mappings in higher dimensions [2].

Let us briefly discuss some technical features of the problem. Recall that for minimizers of the basic integral functional

$$\int_\Omega f\big(Dv(x)\big)\,\mathrm{d}x,$$

where $f = f(z)$ satisfies (2), a classical result states that

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - 2, \quad n \geqslant 3. \tag{6}$$

The singular set is empty in the case $n = 2$. The derivation of the bound (6) has two ingredients. First, the minimizer u satisfies the Euler–Lagrange system: $\operatorname{div} f_z(Du) = 0$. Second, since f is C^2 and strongly convex the difference-quotient method can be used to show that solutions are locally in $W^{2,2}$. From this the bound (6) readily follows (see [7]).

In the general context of functionals of the type (1) none of these ingredients is available. First, as remarked above, the functional is not differentiable in general, and consequently the Euler–Lagrange system might not be available. This is a serious obstruction, since all the techniques developed up to now strongly rely on this. But, as mentioned above, even when the Euler–Lagrange system is available, and even assuming *the smoothness of* F, conditions (2) do not ensure that the system, $\operatorname{div} F_z(x, u, Du) = F_y(x, u, Du)$, can be used to get dimensional estimates for the singular set. Indeed, in the favorable case, where F is smooth and $\alpha = 1$, the right-hand side exhibits critical growth $|F_y(x, u, Du)| \leqslant \tilde{L}(|Du|^p + 1)$, so the singular set estimates based on the Euler–Lagrange system are possible only for bounded solutions satisfying an unavoidable smallness assumption on the L^∞ norm of the type $2\tilde{L}\|u\|_{L^\infty} < \nu$, [10]. In contrast, minimizers of general functionals of the type in (1), may be even unbounded [13]. In fact, as in the case of harmonic maps [12], minimality must be used directly to obtain the singular set estimates, whereas an attempt based solely on the Euler–Lagrange system appears hopeless.

Our proof avoids the direct use of the Euler–Lagrange system and is based on a new, variational argument, using directly the minimality of u. As is clear from the above discussion, the main difficulty in the present setting is caused by the explicit dependence of the energy density $F = F(x, y, z)$ on the variable y, even when $\alpha = 1$ and F is smooth. Indeed, the map $u(x)$ in the energy density $F(x, u(x), Du(x))$ acts as a "measurable coefficient", since for each fixed $z \in \mathbb{R}^{nN}$ the function, $x \mapsto F(x, u(x), z)$, is a-priori only little more than measurable. This is the main reason for the appearance of δ in the estimate (4). Though we have no proof or counter-example, it seems likely that, in general, one cannot go much beyond (4).

In case there is no interaction between the variable v and the gradient variable Dv we can obtain a better estimate. Such is the case for integrals of the type

$$\mathcal{F}_1[v] := \int_\Omega \big(f\big(x, Dv(x)\big) + g\big(x, v(x)\big)\big)\,\mathrm{d}x, \tag{7}$$

where f satisfies (2), suitably rewritten for the case without y dependence, while g is only supposed to be bounded and, satisfy

$$x \mapsto g(x, y) \text{ is measurable} \quad \text{and} \quad \big|g(x, y_1) - g(x, y_2)\big| \leqslant \tilde{L} \min\big\{1, |y_1 - y_2|^\alpha\big\},$$

for any $x \in \Omega$ and $y_1, y_2 \in \mathbb{R}^N$. For minimizers u of $\mathcal{F}_1$ we can improve the bound (4) to

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \alpha. \tag{8}$$

It is interesting to compare this result to previous results valid for solutions of non-linear elliptic systems [9,10]. A functional of the type

$$\int_\Omega f\big(x, Dv(x)\big)\,\mathrm{d}x$$

with f as above, admits an Euler–Lagrange system of the form: $\operatorname{div} A(x, Du) = 0$, where, of course, $A(x, Du) \equiv f_z(x, Du)$. Under the above hypotheses with $p = 2$ it can be shown that the vector field A satisfies the Hölder continuity condition

$$\big|A(x_1, z) - A(x_2, z)\big| \leqslant L|x_1 - x_2|^\beta\big(1 + |z|\big) \quad \text{with } \beta := \alpha/2.$$

Consequently, half of the Hölder continuity exponent is lost when passing from $f(x, z)$ to the partial gradient vector field $f_z(x, z)$, and it is known that this result is sharp. The result in [9] is therefore applicable to the Euler–Lagrange system, yielding the estimate $\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - 2\beta = n - \alpha$, which is in exact accordance with (8). We also observe that for solutions to more general systems of the type $\operatorname{div} A(x, u, Du) = B(x, u, Du)$, the estimate in [10] resembles (4), namely $\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \delta$, where δ depends on the eigenvalues of the matrix A_z. Again this is caused by the explicit dependence of A on u. Summarizing, when there is no Euler–Lagrange system the techniques in [9,10] cannot be applied, however, when the Euler–Lagrange system exists and can be used, the resulting bounds on the Hausdorff dimension are in accordance with the ones in (4) and (8) (see also (12), below).

An intermediate result, between (4) and (8), is also available. It is obtained when the function g above depends on Dv, but has a lower order of growth. Examples are functionals that allow a splitting such as

$$\mathcal{F}_2[v] := \int_\Omega \big(f\big(x, Dv(x)\big) + g\big(x, v(x), Dv(x)\big)\big)\,\mathrm{d}x, \tag{9}$$

where f is as above (but independent of y), g is C^2, convex with respect to the gradient variable and satisfies

$$0 \leqslant g(x, y, z) \leqslant \tilde{L}\big(|z|^\gamma + 1\big) \tag{10}$$

and

$$\big|g(x_1, y_1, z) - g(x_2, y_2, z)\big| \leqslant \tilde{L}\omega\big(|x_1 - x_2| + |y_1 - y_2|\big)\big(|z|^\gamma + 1\big), \quad 0 \leqslant \gamma \leqslant p. \tag{11}$$

A typical model functional is

$$v \mapsto \int_{\Omega} \big(f\big(x, Dv(x)\big) + a\big(x, v(x)\big)\big(1 + \big|Dv(x)\big|^2\big)^{\gamma/2}\big)\,\mathrm{d}x,$$

where $a : \Omega \times \mathbb{R}^n \to \mathbb{R}^+$ is a bounded, α-Hölder continuous function. In these functionals, the presence of u affects the leading term Du in a less severe way, since g grows slower than f in the gradient variable. For minimizers u of $\mathcal{F}_2$ we obtain

$$\dim_{\mathcal{H}}(\Sigma_u) \leqslant n - \min\{\alpha, p - \gamma + \delta\}, \tag{12}$$

where $\delta > 0$ is as in the Theorem 1.1, and thus especially depends on the growth-ellipticity ratio L/ν. The previous bound is intermediate between (4) and (8): (12) reduces to (4) for $\gamma = p$ and to (8) for $\gamma = 0$. In particular, when assuming $\gamma \leqslant p - 1$ we always get the bound in (8).

Finally, we can also prove that under the low dimension assumption

$$n \leqslant p + 2 \tag{13}$$

the estimate in (8) is valid for general functionals of the type (1) under the assumptions (2). This is because in this case one can adapt arguments of Campanato (see [3]) to show that minimizers are Hölder continuous on large sets.

2. Ideas of the proof

For the sake of clarity, we confine the description to the case $p = 2$; we shall only briefly indicate the main points. The details will appear in [8]. The starting point for estimating the Hausdorff dimension of the singular set Σ_u is the well-known inclusion

$$\Sigma_u \subset \left\{ x \in \Omega : \liminf_{\rho \searrow 0} \fint_{B(x,\rho)} \big|Du(y) - (Du)_{x,\rho}\big|^2\,\mathrm{d}y > 0 \text{ or } \limsup_{\rho \searrow 0}\big(|u_{x,\rho}| + \big|(Du)_{x,\rho}\big|\big) = \infty \right\}.$$

This inclusion is standard and is a consequence of all partial regularity proofs [7]. Hence, essentially it suffices to estimate the Hausdorff dimension of the set of non-L^2-Lebesgue points of Du. In turn this is achieved by showing that the gradient of a minimizer belongs to a fractional Sobolev–Slobodetskij space

$$Du \in W^{\theta,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN}) \tag{14}$$

for some $\theta \equiv \theta(n, N, L/\nu) > 0$, depending on the assumptions we are working with, and then following general arguments for the theory of such spaces. Proving (14) is the main part of the proof. We accomplish it in two steps: the first consists of a careful comparison and covering argument that allows us to deal with fractional difference quotients using directly the minimality of u. For the implementation of this argument it is crucial that Du is locally integrable to a higher power $q > p$. Such local higher integrability follows from Gehring's Lemma. The first step therefore produces a small θ_0 such that $Du \in W^{\theta_0,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^{nN})$. The second step exploits the existence of this additional fractional derivative: it is the starting point for an iteration argument that is based on Caccioppoli and Poincaré type inequalities in the setting of fractional Sobolev–Slobodetskij spaces. This iteration improves the degree of (fractional) differentiability of Du and finally leads to the quantities in the bounds (4), (8) and (12), thus concluding the proof. The proof of the estimate (8) under the low dimensional assumption (13) is more involved and combines the above ideas with additional regularity results.

Acknowledgement

Thanks to Emilio Acerbi and Frank Pacard for useful comments. G.M. was partially supported by MIUR/Cofin via the national project "Calcolo delle Variazioni" (2002) and by INDAM via the project "Mappe p-armoniche e immersioni minimali"; he would like to thank the Mathematical Institute, University of Oxford, for the kind hospitality provided during the work for the paper in March 2004.

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